that we are likely to fail in the attempt to ascertain the number of instances of each case [*i.e.*, the number of white and of black pebbles] by observation. But if it is true that we can finally attain moral certainty by this method[4] . . . then we can determine the number of instances *a posteriori* with almost as great accuracy as if they were known to us *a priori*. Axiom 9 [presented in an earlier chapter] shows that in our everyday lives, where moral certainty is regarded as absolute certainty, this consideration enables us to make a prediction about any event involving chance that will be no less scientific than the predictions made in games of chance. If, instead of the jar, for instance, we take the atmosphere or the human body, which conceal within themselves a multitude of the most varied processes or diseases, just as the jar conceals the pebbles, then for these also we shall be able to determine by observation how much more frequently one event will occur than another.

Lest this matter be imperfectly understood, it should be noted that the ratio reflecting the actual relationship between the numbers of the cases—the ratio we are seeking to determine through observation—can never be obtained with absolute accuracy; for if this were possible, the ruling principle would be opposite to what I have asserted: that is, the more observations were made, the *smaller* the probability that we had found the correct ratio. The ratio we arrive at is only approximate: it must be defined by two limits, but these limits can be made to approach each other as closely as we wish. In the example of the jar and the pebbles, if we take two ratios, 301/200 and 299/200, 3001/2000 and 2999/2000, or any two similar ratios of which one is slightly less than 1½ and the other slightly more, it is evident that we can attain any desired degree of probability that the ratio found by our many repeated observations will lie between these limits of the ratio 1½, rather than outside them.

It is this problem that I decided to publish here, after having meditated on it for twenty years. . . .

. . . If all events from now through eternity were continually observed (whereby probability would ultimately become certainty), it would be found that everything in the world occurs for definite reasons and in definite conformity with law, and that hence we are constrained, even for things that may seem quite accidental, to assume a certain necessity and, as it were, fatefulness. For all I know that is what Plato had in mind when, in the doctrine of the universal cycle, he maintained that after the passage of countless centuries everything would return to its original state.

NOTES

1. Translated from *Klassische Stücke der Mathematik*, selected by A. Speiser (Zürich, 1925), pp. 90-95. The selection is from the German translation of the *Ars Conjectandi* by R. Haussner in Ostwald's *Klassiker der exakten Wissenschaften*, Leipzig, 1899, nr. 108.
2. For "case," the correct translation of the German, one may read *result* or *outcome*.
3. *La logique, ou L'art de penser*, by Antoine Arnauld and Pierre Nicole, 1662. (Makes use of Pascal, Fragment no. 14.) There are in fact *two* authors but Bernoulli makes it appear there is only one.
4. Bernoulli demonstrates that this is true in his next chapter.

Chapter VI
The Scientific Revolution at Its Zenith (1620–1720)

Section D
The Bernoullis

JOHANN BERNOULLI (1667–1748)

Johann Bernoulli was the tenth child in his family. His father, Nikolaus, attempted to draw him into the family business. After an unsuccessful apprenticeship as a salesman, however, he received permission from his father to enroll at the University of Basel in 1683. He resided with his brother Jakob. At age 18, Johann received the master of arts degree. At his father's urging, he took up the study of medicine but privately studied mathematics and experimental physics with his brother. He went to Paris in 1691, where he participated in the mathematical circle of Nicolas Malebranche (then the foremost Cartesian). In their discussions Bernoulli disseminated Leibniz's calculus. He also gave calculus lessons to Guillaume-François-Antoine de L'Hospital—lessons which became the basis for the first textbook on the differential calculus, L'Hospital's *Analyse des infiniment petits* ("Analysis of the Infinitely Small," 1696). This text contained the method for evaluating the indeterminate form 0/0, which is incorrectly known today as L'Hospital's rule. In 1693, Bernoulli began an extensive correspondence with Leibniz.

Through the intervention of Christiaan Huygens, Johann Bernoulli was offered the professorial chair in mathematics at Gröningen (Holland) in 1695. He accepted because his quarrels with his brother were growing and because he could not hope to obtain the mathematics professorship at Basel as long as Jakob lived. In September 1695 he, his wife Dorothea Falkner, and their seven-month-old son Nikolaus left for Holland. Two sons were born later—Daniel, the most famous of the Bernoullis, and Johann II. While at Gröningen he did not curb his "Flemish pugnacity." The theologians with whom he argued about natural philosophy charged him with Spinozism, a late-17th-century word for atheism.

Upon the death of Jakob in 1705, Johann succeeded him at the University of Basel. Johann would have preferred to accept other offers extended to him by the Universities of Leiden and Utrecht, but family concerns drew him to his native city where he spent the rest of his life. He was the most distinguished member of the university faculty. In the early 1720s, he taught his greatest student, Leonard Euler. Euler was to be one of his two heroes; the other was Leibniz. His activities were not limited to university affairs, as a member of the Basel school board, he worked to reform its humanistic Gymnasium.

Influential far beyond Switzerland, Johann took part in two major continental scientific quarrels. The Royal Society of London had proclaimed Leibniz a plagiarist in 1712, and Bernoulli was outraged when this news reached Switzerland. Already a strong supporter of Leibniz, he now became his outspoken champion against British critics. Neither Brook Taylor nor Colin Maclaurin, leading British Newtonians, were fully Bernoulli's equal in debates. Bernoulli attacked Newton's method of fluxions and in challenge problems showed the superiority of Leibniz's differential calculus.

On the Continent, the developing Newtonian dynamics and optics began to supplant the dominant Cartesian science during the 1720s and 1730s. The Paris Academy of Sciences was the focal point for the struggle between the two. Once again Bernoulli was an adversary of Newton's ideas, which he considered unduly narrow, especially in comparison to Leibniz's. The chief controversy was over the Cartesian vortex theory and its explanation of celestial motions by whirlpools of the ether—a theory Newton had criticized in Book II of the *Principia*. Bernoulli advocated a revised vortex theory in the papers he submitted to the Paris Academy for its prestigious, biennial prize competition. He won three times—for papers on the transmission of momentum (1727), the motions of the planets in aphelion (1730), and the inclination of the planetary orbits toward the solar equator (1734)—and shared the 1734 prize with his son Daniel who took a Newtonian position. It is said that the quarrelsome and obstinate Johann so begrudged Daniel his position and share of success that he ordered him out of the family house. Johann's work delayed but did not prevent the triumph of Newtonian science that occurred at the Paris Academy between 1734 and 1740.

The relations between Johann Bernoulli and his son Daniel did not improve greatly after 1734; their work in physics was a continuing source of antagonism. Johann advanced continuum mechanics and wrote *Hydraulica* (1738), which made some progress in the study of the internal pressure exerted by fluids on tubes. His book, however, was overshadowed by Daniel's classic *Hydrodynamica* (1738)—a term the son introduced.

The most significant contributions of Johann Bernoulli were to mathematics. He at first worked closely with his brother on probability and the differential calculus. Johann's results appeared in memoirs in *Acta Eruditorum* and the *Journal des Sçavans*. *Acta Eruditorum* for 1694 contained his discovery of the series now known as the Taylor series; the 1697 volume had his skillful paper on the calculus of variations. He was the first to realize that variational problems involve making a given integral a maximum or minimum. His pioneering efforts in the calculus of variations were to be superceded by his pupil Euler. In elaborating the calculus, Johann Bernoulli concentrated on the integration of differential equations; indeed, he defined the integral. From his study of the exponential function, x^y, he derived the natural exponential function, e^x, which is equal to its derivative. Subsequently, Euler made e the base for natural logarithms. Johann and his sons extended the calculus to cover two and three independent variables and provided a flexible formalism to handle the additional degrees of freedom.

80. From "The Curvature of a Ray in Nonuniform Media" (1697)*

(The Brachistochrone)

JOHANN BERNOULLI

The curvature of a ray in nonuniform media, and the solution of the problem to find the brachystochrone, that is, the curve on which a heavy point falls from a given position to another given position in the shortest time, as well as on the construction of the synchrone or the wave of the rays.

. . . We have a just admiration for Huygens, because he was the first to discover that a heavy point on an ordinary *Cycloid* falls in the same time [*tautochronos*], whatever the position from which the motion begins.[1] But the reader will be greatly amazed [*an non obstupescus plane*], when I say that exactly this *Cycloid*, or *Tautochrone of Huygens*, is our required *Brachystochrone*. I reached this understanding in two ways, one indirect and one direct. When I pursued the first, I discovered a wondrous agreement between the curved path of a light ray in a continuously varying medium and our *Brachystochrone*. I also found other rather mysterious things [*in quibus nescio quid arcani subest*] which might be useful in dioptric investigations. It is therefore true, as I claimed when I proposed the problem, that *it is not just naked speculation*, but *also very useful for other branches of knowledge*, namely, for dioptrics. But in order to confirm my words by the deed, let me here give the first mode of proof!

Fermat, in a letter to De la Chambre,[2] has shown that a light ray passing from a thin to a more dense medium, is bent toward the perpendicular in such a way that, under the supposition that the ray moves continuously from the light to the illuminated point, it follows the path that requires the shortest time. With the aid of these principles he showed that the sine of the angle of incidence and the sine of the angle of refraction are in inverse proportion to the densities of the media, hence directly as the velocities with which the light ray penetrates these media. Later Leibniz, in the *Acta Eruditorum*, 1682, pp. 185 sequ., and soon afterward the famous Huygens in his *Treatise on Light*, p. 40,[3] have demonstrated this more comprehensively and, by most valid arguments, have established the physical, or better the metaphysical, principle which Fermat seems to have abandoned at the insistence of Clerselier, remaining satisfied with his geometric proof and giving up his rights all too lightly.

Now we shall consider a medium that is not homogeneously dense, but consists of purely parallel horizontally superimposed layers, of which each consists of diaphanous matter of a certain density decreasing or increasing according to a certain law. It is then manifest that a ray which we consider as a particle will not be propagated in a

*Source: This translation of "Curvatura radii in diaphanis non uniformibus" is taken from D. J. Struik (ed.), *A Source Book in Mathematics, 1200-1800* (1969), 392-396. Reprinted by permission of Harvard University Press, straight line, but in a curved path. This has already been considered by Huygens in his above-mentioned *Treatise on Light*, but he did not determine the nature of this minimizing curve such that the particle, whose velocity increases and decreases depending on the density of the medium, will pass from point to point in the shortest time. We know that the sines of the angles of refraction at the separate points are to each other inversely as the densities of the media or directly as the velocities of the particles, so that the brachystochrone curve has the property that the sines of its angles of inclination with respect to the vertical are everywhere proportional to the velocities. But now we see immediately that the brachystochrone is the curve that a light ray would follow on its way through a medium whose density is inversely proportional to the velocity that a heavy body acquires during its fall. Indeed, whether the increase of the velocity depends on the constitution of a more or less resisting medium, or whether we forget about the medium and suppose that the acceleration is generated by another cause according to the same law as that of gravity, in both cases the curve is traversed in the shortest time. Who prohibits us from replacing the one by the other?

In this way we can solve the problem of an arbitrary law of acceleration, since it is reduced to the determination of the path of a light ray through a medium of arbitrarily varying density. Hence let *FCD* [Fig. 1] be the medium bounded by the horizontal line *FG* on which the luminous point *A* is situated. Let the curve *AHE*, with vertical axis *AD*, be given, its ordinates *HC* determining the densities of the medium at altitude *AC* or the velocities of the light rays or particles at *M*. Let the curved line of that light ray, which we wish to determine, be *ABM*. Let us write for *AC*, *x*; for *CH*, *t*; for *CM*, *y*; and for the differentials *Cc*, *dx*; diff. *mn* = *dy*; diff. *Mm* = *dz*, finally, let *a* be an arbitrary constant. Then *Mm* is the total sine, *mn* the sine of the angle of refraction or the angle of inclination of the curve with respect to the vertical. As we have said before, the ratio of *mn* to *CH* is constant, hence

$$dy : t = dz : a,$$

so that

$$a\, dy = t\, dz,$$

or

$$aa\, dy^2 = tt\, dz^2 = tt\, dx^2 + tt\, dy^2.$$

This gives a general differential equation for the required curve *ABM*:

$$dy = t\, dx : \sqrt{(aa - tt)}.$$

In this way I have solved at one stroke two important problems—an optical and a mechanical one—and have achieved more than I have demanded from others: I have shown that two problems, taken from entirely separate fields of mathematics, have the same character.

Now let us take a special case, namely the common hypothesis first in-

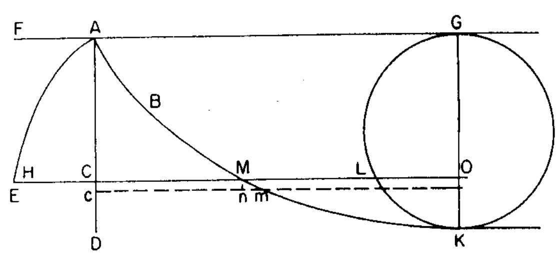

FIG. 1

troduced by Galilei, who proved that the velocities of falling bodies are to each other as the square roots [*in ratione subduplicata*] of the altitudes traversed—then this is really the given problem. Under this assumption the given curve *AHE* is a parabola $tt = ax$, hence $t = \sqrt{ax}$. If this value is substituted in the original equation, we obtain

$$dy = dx \sqrt{\frac{x}{a-x}},$$

from which I conclude that the *Brachystochrone* is the ordinary *Cycloid*. For when the circle *GLK* of radius *a* rolls on *AG* and the rolling starts at *A*, the point *K* describes a cycloid, of which the differential equation is exactly

$$dy = dx \sqrt{\frac{x}{a-x}},$$

if $AC = x$, $CM = y$.

Bernoulli then shows this analytically by writing

$$dx \sqrt{\frac{x}{a-x}} = \frac{1}{2}\frac{a\,dx}{\sqrt{ax-x^2}} - \frac{1}{2}\frac{a\,dx - 2x\,dx}{\sqrt{ax-x^2}},$$

which integrated gives

$$CM = \text{arc } GL - LO,$$

from which, since $MO = CO - \text{arc } GL + LO = \text{arc } LK + LO$, it follows that $ML = \text{arc } LK$.[4]

To solve the problem completely he then shows that from a given point as vertex a cycloid can be described that passes through a second given point.

Before I end I must voice once more the admiration that I feel for the unexpected identity of Huygens' tautochrone and my brachystochrone. I consider it especially remarkable that this coincidence can take place only under the hypothesis of Galilei, so that we even obtain from this a proof of its correctness. Nature always tends to act in the simplest way, and so it here lets one curve serve two different functions, while under any other hypothesis we should need two curves, one for tautochronic oscillations, the other for the most rapid fall. If, for example, the velocities were as the altitudes, then both curves would be algebraic, the one a circle, the other one a straight line.[5]

Bernoulli then introduces the *synchrone*: the curve *PB* [Fig. 2] in a vertical plane such that a heavy body falling from *A* along this curve reaches the points *B* in the same time as a heavy body falling on the cycloid *AB*. Referring to Huygens, he concludes that *PB* is also a cycloid intersecting all cycloids with initial point *A* at a right angle. He ends by suggesting that other orthogonal trajectories of given families of curves be found.[6]

NOTES

1. Huygens, *Horologium oscillatorium* (Paris, 1673), Proposition XXV: In a cycloid with vertical axis and with its vertex down,

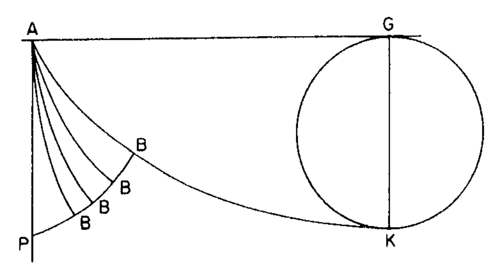

FIG. 2

the times of descent in which a mobile particle, starting from rest at an arbitrary point of the curve, reaches the lowest point are equal among themselves, and have to the time of the vertical fall along the total axis of the cycloid a ratio equal to that of the semicircumference of a circle to its diameter. *Oeuvres complètes*, XVIII (1934), 185.

2. Fermat's letters to Martin Cureau de la Chambre are of 1657 and 1662 (*Oeuvres*, II, 354-359, 457-463). The law of refraction was published by Descartes in his *Dioptrique* (1637). Fermat first opposed it, but then reestablished it by a maximum-minimum principle.

3. Huygens, *Traité de la lumière* (Leiden, 1690), 40; *Oeuvres complètes*, XIX (1737), 489.

4. This gives the equation of the cycloid in the form

$$x = \frac{a}{2}(1 - \cos t), \quad y = \frac{a}{2}(t - \sin t),$$

$$t = \pi - \phi, \quad \text{arc } LK = a\phi.$$

The differential equation can already be found in Leibniz's first paper on the integral calculus of 1686.

5. The cases mentioned are $t = ax$ and $t = ax^{1/3}$.

6. Johann Bernoulli does not yet use the term "orthogonal trajectories." The concept played an important role in the work of Leibniz and Bernoulli in those days. The connection with Huygens's theory of light was clear. The term "trajectory" dates from an article by Johann Bernoulli in the *Acta Eruditorum* of 1698 (*Opera omnia*, I, 266).

Chapter VII

The Age of Enlightenment and the French Revolution (1720-1800)

Introduction. The spectacular development of European mathematics in the hands of a few masters continued throughout the 18th century. Above all, the methods of the differential and the fluxional calculus were combined to form the unified mathematical discipline of analysis, which was then greatly extended. This was largely the result of the work of the Swiss-born Leonhard Euler, who dominated mathematics. These inventions widened the scope of mathematics and intensified its relations with rational mechanics[1] and astronomy, that is, celestial mechanics, the exact science in which the calculus was applied with the most dramatic results. Attempts to solve special, previously unassailable problems encouraged growth. Studies of the shape of vibrating strings fixed at their end points and of elastic beams under tension led to solutions requiring new principles and analytical methods, and so did successful efforts to describe more accurately lunar motion and the flow of water through pipes. Mathematical analysis was employed with extraordinary success in problem-solving by Daniel Bernoulli, Alexis Clairaut, Jean d'Alembert, Leonhard Euler, Louis Lagrange, and others. Three of its component parts—differential equations,[2] infinite series, and the calculus of variations—quickly developed into distinct areas of inquiry. Mathematical analysis itself not only gained autonomy as a branch of mathematics, forming a triumvirate with geometry and algebra, but also displaced geometry (both Euclidean and more recently Cartesian) from the primacy in mathematics that it had held for two millennia.

Of course 18th-century mathematics had other significant developments. It consisted of far more than the creation and exploitation of analysis. The term "mathematics" was applied to an array of subjects. D'Alembert's "Detailed System of Human Knowledge" given at the conclusion of his *Preliminary Discourse to the Encyclopedia* (1751) accurately maps these. "Pure mathematics" consisted of arithmetic (which included algebra and the calculus) and geometry, while the larger contiguous realm of "mixed mathematics" included the exact sciences of geometric astronomy, optics, acoustics, pneumatics, and rational mechanics, which, in turn, covered the more technological fields of ballistics, navigation and shipbuilding.

The principal achievements in the branches of mathematics outside of analysis were less original and largely motivated by it. In algebra improved methods of solution were devised for polynomial equations and the theory of numbers began to