and subsequent successive orders that vanish be an even number, the ordinate is then neither a *maximum* nor *minimum*.

860. When the fluxion of the ordinate y is supposed equal to nothing, and an equation is thence derived for determining x, if the roots of this equation are all unequal, each gives a value of x that may correspond to a greatest or least ordinate. But if two, or any even number of these roots be equal, the ordinate that corresponds to them is neither a *maximum* nor *minimum*. If an odd number of these roots be equal, there is one *maximum* or *minimum* that corresponds to these roots, and one only. Thus if $\frac{\dot{y}}{\dot{x}} = x^4 + ax^3 + bx^2 + cx + d$, then supposing all the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ to be real, if the four roots are equal there is no ordinate that is a *maximum* or *minimum;* if two or three of the roots only are equal, there are two ordinates that are *maxima* or *minima;* and if all the roots are unequal there are four such ordinates.

861. To give a few examples of the most simple cases. Let $y = a^2x - x^3$, then $\dot{y} = a^2\dot{x} - 3x^2\dot{x}$ and $\ddot{y} = -6x\dot{x}^2$. Suppose $\dot{y} = 0$, and $3x^2 = a^2$ or $x = \frac{a}{\sqrt{3}}$, in which case $\ddot{y} = \frac{-6a\dot{x}^2}{\sqrt{3}}$. Therefore $\ddot{y}$ being negative, y is a *maximum* when $x = \frac{a}{\sqrt{3}}$, and its greatest value is $\frac{2a^3}{3\sqrt{3}}$. If $y = aa + 2bx - xx$, then $\dot{y} = 2b\dot{x} - 2x\dot{x}$, and $\ddot{y} = -2\dot{x}^2$; consequently y is a *maximum* when $2b - 2x = 0$, or $x = b$. If $y = aa - 2bx + xx$ then $\dot{y} = 2b\dot{x} + 2x\dot{x}$, and $\ddot{y} = 2\dot{x}^2$; consequently y is now a *minimum* when $x = b$, if a be greater than b.

[Maclaurin also considers the cases in which $\ddot{y}$, $\dddot{y}$, $\ddddot{y}$, . . . vanish.]

NOTE

1. Maclaurin's book is divided into two parts. Book I is geometrical, Book II is computational. Our selection is from Book II. Articles 255 and 261 (to which he refers below) deal with the same matter in a geometrical way.

JEAN LE ROND D'ALEMBERT (1717–83)

The Frenchman Jean d'Alembert attained a reputation in the mathematical sciences before achieving fame as a *philosophe* during the Continental Enlightenment. In the sciences he substantially advanced mathematical analysis and rational mechanics, and he believed, like Locke and Condillac, that sense perception provides the basic evidence about the physical world. As a *philosophe*, he came to rank just below Voltaire and Denis Diderot, the general editor of the *Encyclopédie* (28 vols., 1751–72).

During the Enlightenment deductive reason was supplanting religious faith as the chief guide to social action among the educated public. D'Alembert maintained that the increased use of reason would lead to progress. He also advocated tolerance, free speech, and enlightened absolutism as well as criticizing established religion.

The illegitimate son of a salon hostess, Madame de Tencin, and a cavalry officer named Destouches-Canon, d'Alembert was abandoned on the steps of the Parisian church Saint Jean-Le-Rond by his mother who had just renounced her nun's vows and may have feared that civil authorities would forcibly return her to a convent if they learned of the birth. The father located the infant and found him a home with a humble glazier, named Rousseau, and his wife. They christened the child Jean le Rond for the church where he was found, and he lived with his adoptive parents until he was 47 years old. His natural father, though he did not reveal his identity, provided an annual annuity of 1200 livres and gained him admission to the prestigious College Mazarin, a Jansenist school which stressed classics and rhetoric. There d'Alembert developed an aversion for religious studies, and turned to law, becoming an advocate in 1738. He then briefly studied medicine before beginning work in the mathematical sciences, which he learned largely by himself. Later he would write that mathematics was "the only occupation which really interested me."

In 1739, d'Alembert submitted his first *memoir* to the Paris Academy of Sciences. During the next two years he submitted five more papers, which dealt with differential equations and with the motion of bodies in resisting media. He made himself familiar with the writings of Newton, L'Hôpital, the Bernoullis, and major contemporary geometers. Following several unsuccessful attempts to gain admittance to the Paris Academy, he was finally elected a member in 1741. After a two-year study of several problems in mechanics, he hastily published his most famous scientific work, *Traité de dynamique* (1743), which helped to formalize the new science of dynamics. The *Traité* contains "d'Alembert's principle," which maintains that Newton's third law of motion (every action has an equal and opposite reaction) holds for moving and rigidly fixed bodies. It also helped to resolve the controversy over the principle of the conservation of *vis viva* (mv^2). In this dispute the Newtonians and Cartesians asserted that the "quantity of motion" *(mv)* gave the correct measure of force in the study of collisions. The followers of Leibniz and Wolff disagreed; they claimed that mv^2 was the correct measure. Pointing out in the preface that Newton's force could be defined either as acting through space ($mv^2 = 2Fs$) or over time ($mv = mat = Ft$), d'Alembert declared this controversy over force measurement to be a false one—a quarrel of words.

By the middle of the 18th century, d'Alembert stood among the leading mathematicians and theoretical physicists in Europe. Three others were his French rival Alexis Clairaut, Daniel Bernoulli in Basel, and Leonhard Euler in Berlin and St. Petersburg, with Euler the most able of the group. In 1744, d'Alembert published a landmark treatise on fluid mechanics, which correctly established that if one assumes the earth to be a rotating fluid body, it must have an orange-like shape. Over the next three years he developed partial differential equations as a branch of the calculus and was the first to generally apply them to problems in physics, including that of the motion of vibrating chords. In 1749, his interest in the three-body problem in celestial mechanics led him to explain the precession of equinoxes—a gradual shift in the position of the earth's orbit—and the nutation or wobbling of the earth's axis. In his essay on hydrodynamics published in 1752, differential hydrodynamic equations were first expressed in terms of a field—a pioneering attempt in complex function theory—and the later discredited "d'Alembert's paradox" was introduced.

After 1750, d'Alembert turned increasingly to interests beyond the sciences, becoming associated with the *Encyclopédie*—the chief intellectual enterprise in Europe in the mid-18th century and the center of opposition to the *Ancien Regime*. He wrote the *Discourse préliminaire* (1751) to the *Encyclopédie* and served as its science editor for seven years. In 1756, he traveled to Geneva to enjoy a leisurely visit with Voltaire and to collect material for an article on the city. What he wrote was a tendentious four-page piece that appeared in the seventh volume of the *Encyclopédie*. In it d'Alembert claimed that some Genevan pastors "no longer believe in the divinity of Jesus Christ," and he praised them for their learning, their freedom from superstition, and their support of theatre. The publication of the article aroused a public furor in both Geneva and Paris, and d'Alembert prudently resigned the science editorship of the *Encyclopédie*. However, his action brought him strained relations with the shaken editor, Diderot, who considered him a deserter. The next year, after vehement public debate, the French government suspended the license of the *Encyclopédie*.

There were other tasks facing d'Alembert. The success of the *Discours préliminaire* and the intercession of Mme. du Deffand, whose home was a prominent salon for literary men and savants, had brought about his acceptance to the French Academy in 1754. He worked zealously to enhance its dignity and was made perceptual secretary in 1772. As his scientific and literary fame spread, foreign monarchs vied for his services. In 1764, he spent three months at Potsdam with Frederick the Great who wanted him to be president of the Berlin Academy. He refused the presidency, however, and recommended Euler for the position. His support for Euler healed a rift that had developed more than a decade earlier when d'Alembert believed that Euler had blocked his winning of a prize from the Berlin Academy for a paper on fluid mechanics. Refusing to leave Paris, the cultural capital of Europe, d'Alembert subsequently declined an offer from Catherine the Great who wanted him to improve the Russian educational system.

A small man with a highly pitched voice, d'Alembert was known in Parisian society for his gaiety, witty conversation, and talent for mimicry. He usually worked both in the morning and afternoon, spending his evenings in the salons where the cultivated public gathered. Practicing frugality, he was satisfied with his limited means. He enjoyed fair health until 1765 when he fell gravely ill. Although he never married, he moved at that time into the house of his lover, Mlle. de Lespinasse, and resided there until her death in 1776. He spent his last years in an apartment at the Louvre.

His contributions to mathematical analysis were extensive. Almost alone at this time he regarded the derivative as the limit of a quotient of increments, or what we now express as dy/dx. Eventually the calculus would be rationalized around the key concept of the limit, but d'Alembert was not able to put it in to a purely algorithmic form. He stressed the law of continuity in analysis and called equations with discontinuities impossible. His continuity requirement probably led him to the idea of a limit and made him examine the techniques for handling infinite series. In volume V of his *Opuscules mathématiques* (8 vols., 1761–80) he published d'Alembert's theorem (the ratio test for convergency). His theorem follows:

If $\lim \left[\frac{S_{n+1}}{S_n}\right] = r$ and $r < 1$, then the series $\sum_{n=1}^{\infty} S_n$ converges.

If $r > 1$, the series diverges; if $r = 1$, the test fails.

In mathematics, he also considered the parallel postulate in Euclidean geometry a "scandal" and worked on probability theory, applying it to games of chance and to determining life expectancy.

84. From "Differential," *Encyclopédie*, Vol. 4 (1754)*[1]

(On Limits)

JEAN d'ALEMBERT

. . What concerns us most here is the metaphysics of the *differential* calculus.

This metaphysics, of which so much has been written, is even more important and perhaps more difficult to explain than the rules of this calculus themselves: various mathematicians, among them Rolle,[2] who were unable to accept the assumption concerning infinitely small quantities, have rejected it entirely, and have held that the principle was false and capable of leading to error. Yet in view of the fact that all results obtained by means of ordinary Geometry can be established similarly and much more easily by means of the *differential* calculus, one cannot help concluding that, since this calculus yields reliable, simple, and exact methods, the principles on which it depends must also be simple and certain.

Leibniz was embarrassed by the objections he felt to exist against infinitely small quantities, as they appear in the *differential* calculus; thus he preferred to reduce infinitely small to merely in-

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comparable quantities. This, however, would ruin the geometric exactness of the calculations; is it possible, said Fontenelle,[3] that the authority of the inventor would outweigh the invention itself? Others, like Nieuwentijt,[4] admitted only *differentials* of the first order and rejected all others of higher order. This is impossible; indeed, considering an infinitely small chord of first order in a circle, the corresponding abscissa or versed sine is infinitely small[5] of second order; and if the chord is of the second order, the abscissa mentioned will be of the fourth order, etc. This is proved easily by elementary geometry, since the diameter of a circle (taken as a finite quantity) is always to the chord as the chord to the corresponding abscissa.[6] Thus, if one admits the infinitely small of the first order, one must admit all the others, though in the end one can rather easily dispense with all this metaphysics of the infinite in the *differential* calculus, as we shall see below.

Newton started out from another principle; and one can say that the metaphysics of this great mathematician on the calculus of fluxions is very exact and illuminating, even though he allowed us only an imperfect glimpse of his thoughts.

He never considered the *differential* calculus as the study of infinitely small quantities, but as the method of first and ultimate ratios, that is to say, the method of finding the limits of ratios. Thus this famous author has never differentiated quantities but only equations; in fact, every equation involves a relation between two variables and the differentiation of equations consists merely in finding the limit of the ratio of the finite differences of the two quantities contained in the equation. Let us illustrate this by an example which will yield the clearest idea as well as the most exact description of the method of the *differential* calculus.

Let *AM* [Fig. 1] be an ordinary parabola, the equation of which is $yy = ax$; here we assume that $AP = x$

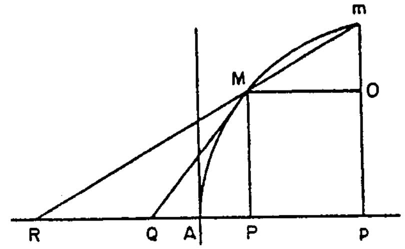

FIG. 1

and $PM = y$, and a is a parameter. Let us draw the tangent MQ to this parabola at the point M. Let us suppose that the problem is solved and let us take an ordinate pm at any finite distance from PM; furthermore, let us draw the line mMR through the points M, m. It is evident, *first,* that the ratio[7] MP/PQ of the ordinate to the subtangent is greater than the ratio MP/PR or mO/MO which is equal to it because of the similarity of the triangles MOm, MPR; *second,* that the closer the point m is to the point M, the closer will be the point R to the point Q, consequently the closer will be the ratio MP/PR or mO/MO to the ratio MP/PQ; finally, that the first of these ratios approaches the second one as closely as we please, since PR may differ as little as we please from PQ. Therefore, the ratio MP/PQ is the limit of the ratio of mO to OM. Thus, if we are able to represent the ratio mO/OM in algebraic form, then we shall have the algebraic expression of the ratio MP to PQ and consequently the algebraic representation of the ratio of the ordinate to the subtangent, which will enable us to find this subtangent. Let now $MO = u$, $Om = z$; we shall have $ax = yy$, and $ax + au = yy + 2yz + zz$. Then in view of $ax = yy$ it follows that $au = 2yx + zz$ and $z/u = a/(2y + z)$.

This value $a/(2y + z)$ is, therefore, in general the ratio of mO to OM, wherever one may choose the point m. This ratio is always smaller than $a/2y$; but the smaller z is, the greater the ratio will be and, since one may choose z as small as one pleases, the ratio $a/(2y + z)$ can be

brought as close to the ratio $a/2y$ as we like. Consequently $a/2y$ is the limit of the ratio $a/(2y + z)$, that is to say, of the ratio mO/OM. Hence $a/2y$ is equal to the ratio MP/PQ, which we have found to be also the limit of the ratio of mO to Om, since two quantities that are the limits of the same quantity are necessarily equal to each other. To prove this, let X and Z be the limits of the same quantity Y. Then I say that $X = Z$; indeed, if they were to have the difference V, let $X = Z \pm V$: by hypothesis the quantity Y may approach X as closely as one may wish; that is to say, the difference between Y and X may be as small as one may wish. But, since Z differs from X by the quantity V, it follows that Y cannot approach Z closer than the quantity V and consequently Z would not be the limit of Y, which is contrary to the hypothesis.

From this it follows that MP/PQ is equal to $a/2y$. Hence $PQ = 2yy/a = 2x$. Now, according to the method of the *differential* calculus, the ratio of MP to PQ is equal to that of dy to dx; and the equation $ax = yy$ yields $a\,dx = 2y\,dy$ and $dy/dx = a/2y$. So dy/dx is the limit of the ratio of z to u, and this limit is found by making $z = 0$ in the fraction $a/(2y + z)$.

But, one may say, is it not necessary also to make $z = 0$ and $u = 0$ in the fraction $z/u = a/(2y + z)$, which would yield $0/0 = a/2y$? What does this mean? My answer is as follows. First, there is no absurdity involved; indeed $0/0$ may be equal to any quantity one may wish: thus it may be $= a/2y$. Secondly, although the limit of the ratio of z to u has been found when $z = 0$ and $u = 0$, this limit is in fact not the ratio of $z = 0$ to $u = 0$, because the latter one is not clearly defined; one does not know what is the ratio of two quantities that are both zero. This limit is the quantity to which the ratio z/u approaches more and more closely if we suppose z and u to be real and decreasing. Nothing is clearer than this; one may apply this idea to an infinity of other cases.[8]

Following the method of differentiation (which opens the treatise on the quadrature of curves by the great mathematician Newton), instead of the equation $ax + au = yy + 2yz + zz$ we might write $ax + a0 = yy + 2y0 + 00$, thus, so to speak, considering z and u equal to zero; this would have yielded $0/0 = a/2y$. What we have said above indicates both the advantage and the inconveniences of this notation: the advantage is that z, being equal to 0, disappears without any other assumption from the ratio $a/(2y + 0)$; the inconvenience is that the two terms of the ratio are supposed to be equal to zero, which at first glance does not present a very clear idea.

From all that has been said we see that the method of the *differential* calculus offers us exactly the same ratio that has been given by the preceding calculation. It will be the same with other more complicated examples. This should be sufficient to give beginners an understanding of the true metaphysics of the *differential* calculus. Once this is well understood, one will feel that the assumption made concerning infinitely small quantities serves only to abbreviate and simplify the reasoning; but that the *differential* calculus does not necessarily suppose the existence of those quantities; and that moreover this calculus merely consists in *algebraically determining the limit of a ratio, for which we already have the expression in terms of lines, and in equating those two expressions. This will provide us with one of the lines we are looking for.* This is perhaps the most precise and neatest possible definition of the *differential* calculus; but it can be understood only when one is well acquainted with this calculus, because often the true nature of a science can be understood only by those who have studied this science.

In the preceding example the known geometric limit of the ratio of z to u is the ratio of the ordinate to the subtangent; in the *differential* calculus we look for the algebraic limit of the ratio z to u and we find $a/2y$. Then, calling s the subtangent, one has $y/s = a/2y$; hence $s = 2yy/a = 2x$. This example is sufficient to understand the others. It will, therefore, be sufficient to make oneself familiar with the previous example concerning the tangents of the parabola, and, since the whole *differential* calculus can be reduced to the problem of the tangents, it follows that one could always apply the preceding principles to various problems of this calculus, for instance to find *maxima and minima*, points of inflection, cusps, etc. . . .[9]

What does it mean, in fact, to find a maximum or a minimum? It consists, it is said, in setting the difference[10] dy equal to zero or to infinity; but it is more precise to say that it means to look for the quantity dy/dx which expresses the limit of the ratio of finite dy to finite dx, and to make this quantity zero or infinite. In this way all the mystery is explained; it is not dy that one makes = to infinity: that would be absurd, since dy is taken as infinitely small and hence cannot be infinite; it is dy/dx: that is to say, one looks for the value of x that renders the limit of the ratio of finite dy to finite dx infinite.

We have seen above that in the *differential* calculus there are really no infinitely small quantities of the first order; that actually those quantities called u are supposed to be divided by other supposedly infinitely small quantities; in this state they do not denote either infinitely small quantities or quotients of infinitely small quantities; they are the limits of the ratio of two finite quantities. The same holds for the second-order differences and for those of higher order. There is actually no quantity in Geometry such as $d\ dy$; whenever $d\ dy$ occurs in an equation it is supposed to be divided by a quantity dx^2, or another of the same order. What now is $d\ dy/dx^2$? It is the limit of the ratio $d\ dy/dx$ divided by dx; or, what is still clearer, it is the limit of dz/dx, where $dy/dx = z$ is a finite quantity.

NOTES

1. (Editor's Note: For an *histoire du livre* of the *Encyclopédie*, one should consult Robert Darnton, *The Business of Enlightenment: A Publishing History of the Encyclopédie, 1775–1800* (Cambridge, Mass.: Harvard University Press, 1979).)

2. Michel Rolle (1652–1719), member of the Paris Academy, is best known for the theorem in the theory of equations called after him. In 1700 he took part in a debate in the Paris Academy on the principles of the calculus; see C. Boyer, *The History of the Calculus* (Dover, New York, 1949), 241.

3. Bernard le Bovier de Fontenelle (1657–1757) was a predecessor of d'Alembert as *secrétaire perpétuel* of the Academy. See Boyer, *History*, 241-242.

4. Bernard Nieuwentijt (1654–1718), a physician-burgomaster of Purmerend, near Amsterdam, opposed Leibniz's concept of the calculus.

5. Versed $\sin \alpha = 1 - \cos \alpha = \alpha^2/2! - \alpha^4/4! + \ldots$ (d'Alembert still takes the dimension to be that of a chord, hence his vers α is really our R vers α).

6. $2R : 2R \sin \alpha/2 = 2R \sin \alpha/2 : R(1 - \cos \alpha)$.

7. D'Alembert writes $\frac{MP}{PQ}$.

8. Here d'Alembert refers to his articles on "Limit" and "Exhaustion" in the same *Encyclopédie*.

9. Here d'Alembert refers to his articles on these subjects.

10. D'Alembert makes little distinction between *différence* and *différentiel*.

LEONHARD EULER (1707–83)

Leonhard Euler was one of the two leading figures in the exact sciences in the 18th century. During that century, only the Savoyard Louis Lagrange compared with him in brilliance and achievement in mathematics and theoretical physics; no one in these fields compared with him in his prolific writing. Euler swiftly and clearly wrote over 866 books and articles, which constitute about one third of the entire corpus published between 1725 and 1800 on the subjects of mathematics, theoretical physics, and engineering mechanics. His publications fill 74 quarto volumes of 300 to 600 pages each. He also engaged in an extensive correspondence, exchanging as many as 5,000 letters with scientists, administrators, and savants across Europe. His letters, many of which are like articles in a modern research journal, cover a wide range of topics, including architecture, biology, chemical science, history, philosophy, religion, and technology.

Euler was born in Basel, Switzerland. His father, Paul, was a Zwinglian minister; his mother, Margaret Brucker, was the daughter of another minister. He grew up in the Swiss countryside in Riehen in a two-room parsonage with two younger sisters. At home, his mother instructed Leonhard in classical humanities, and his father, who had studied under Jakob Bernoulli, taught him mathematics and religion. As a child he developed the forthright disposition and deep religious conviction for which he was known lifelong.

In 1719, Euler was sent to Basel's humanistic Gymnasium for formal schooling and one year later enrolled at the University of Basel, where he displayed keen abilities and was graduated with first honors in 1722. He also revealed a phenomenal memory by reciting Vergil's *Aeneid* by heart. At his father's bidding in 1723, he began theological studies at the university in preparation for the ministry. However, he was already deeply interested in mathematics and, with effort, convinced the stern and difficult Johann Bernoulli to tutor him in mathematics and natural philosophy for one hour on Saturday afternoons. He read classics in these fields and presented problems that he could not solve. Bernoulli quickly recognized the boy's genius and helped to convince Paul Euler to allow his son to concentrate on the mathematical sciences.

In 1727, after failing to obtain a physics position in Basel, Euler joined the St. Petersburg Academy of Sciences. When the Russian government stopped its funds, he served as a medical lieutenant in the Russian navy from 1727 to 1730. He became professor of natural philosophy at the Academy in 1730 and first professor of mathematics (the premier post) in 1733 succeeding Daniel Bernoulli, who returned to Switzerland. Until then he had boarded at Daniel Bernoulli's home. Among the topics the two men discussed at dinner was Bernoulli's book *Hydrodynamica* (1738). In December 1733, Euler married Catherine Gsell, the daughter of a Dutch artist living in Russia. They had 13 children, five of whom survived childhood.

In the years from 1733 to 1741, Euler immersed himself in research on