COMMENTARY BY D. J. STRUIK

Fermat then extends his method to parabolas. His reasoning can be translated as follows.

Divide the interval $0 \leqslant x < a$ into parts by the points $x_1 = a$, $x_2 = ar$, $x_3 = ar^2, \ldots, r < 1$, which are separated by the intervals $l_1 = a(1 - r)$, $l_2 = ar(1 - r)$, $l_3 = ar^2(e - r), \ldots$. If $y = x^n$ ($n = p/q$, p, $q \lessgtr 0$) is the equation of the "hyperbola" or "parabola," then the values of y corresponding to $x_1, x_2, x_3, \ldots$ are $y_1 = a^n$, $y_2 = a^n r^n$, $y_3 = a^n r^{2n}, \ldots$. Then the sum S of the rectangles $l_1x_1 + l_2x_2 + l_3x_3 + \cdots$ is

$$S = a(1-r)a^n + ar(1-r)a^n r^n + ar^2(1-r)a^n r^{2n} + \cdots$$

$$= (1-r)a^{n+1}(1 + r^{n+1} + r^{2n+2} + \cdots)$$

$$= \frac{1-r}{1-r^{n+1}}a^{n+1}.$$

When $r = s^q$ ($s < 1$) and $n \neq -1$, then

$$\int_0^a x^n\,dx = a^{n+1}\lim \frac{1-r}{1-r^{n+1}} = a^{n+1}\lim \frac{1-s^q}{1-s^{p+q}}$$

$$= \frac{qa^{n+1}}{p+q} = \frac{a^{n+1}}{n+1}.$$

As we see, this procedure holds for n positive and negative, but it fails for $n = -1$.

This method approaches our modern method of limits; it uses the concept of the limit of an infinite geometric series.

NOTES

1. *Editor's note.* This paper generalizes Bonaventura Cavalieri's integral, which in modern symbols is $\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$ where n is positive whole number, to n fractional or negative.
2. Fermat uses the Greek term *tetragonizein* for "to perform a quadrature," a practice not uncommon in the seventeenth century.
3. This is Fermat's way of expressing that the sum of a convergent series $a + ar + ar^2 + \cdots + ar^n + \cdots = a/(1 - r)$.
4. This may mean "exponents that are unit fractions."
5. The hyperbola of Apollonius is the ordinary hyperbola, of which, if its equation is $xy = a^2$, the integral $\int_0^\infty y\,dx$ diverges.
6. The term *adequatio* is a Latin translation of the Greek term *parisótēs,* by which Diophantus denoted an approximation to a certain number as closely as possible. See T. L. Heath, *Manual of Greek Mathematics* (Clarendon Press, Oxford, 1931), 493. Fermat uses the term to denote what we call a limiting process.

70. From "On a Method for the Evaluation of Maxima and Minima"[1]*

(Fermat obtained a general method to find the extrema of a given function. His algorithm was subsequently developed into the method of the "characteristic triangle," *dx*, *dy*, and *ds*.)

PIERRE FERMAT

The whole theory of evaluation of maxima and minima presupposes two unknown quantities and the following rule:

Let *a* be any unknown of the problem (which is in one, two, or three dimensions, depending on the formulation of the problem). Let us indicate the maximum or minimum by *a* in terms which could be of any degree. We shall now replace the original unknown *a* by *a* + e and we shall express thus the maximum or minimum quantity in terms of *a* and e involving any degree. We shall adequate [*adégaler*], to use Diophantus' term,[2] the two expressions of the maximum or minimum quantity and we shall take out their common terms. Now it turns out that both sides will contain terms in e or its powers. We shall divide all terms by e, or by a higher power of e, so that e will be completely removed from at least one of the terms. We suppress then all the terms in which e or one of its powers will still appear, and we shall equate the others; or, if one of the expressions vanishes, we shall equate, which is the same thing, the positive and negative terms. The solution of this last equation will yield the value of *a*, which will lead to the maximum or minimum, by using again the original expression.

Here is an example:

To divide the segment AC [Fig. 1] *at E so that AE × EC may be a maximum.*

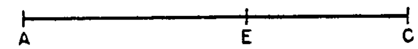

FIG. 1

We write $AC = b$; let a be one of the segments, so that the other will be $b - a$, and the product, the maximum of which is to be found, will be $ba - a^2$. Let now $a + e$ be the first segment of b; the second will be $b - a - e$, and the product of the segments, $ba - a^2 + be - 2ae - e^2$; this must be adequated with the preceding: $ba - a^2$. Suppressing common terms: $be \sim 2ae + e$. Suppressing e: $b = 2a$.[3] To solve the problem we must consequently take the half of b.

We can hardly expect a more general method.

ON THE TANGENTS OF CURVES

We use the preceding method in order to find the tangent at a given point of a curve.

Let us consider, for example, the parabola *BDN* [Fig. 2] with vertex *D* and of diameter *DC*; let *B* be a point on

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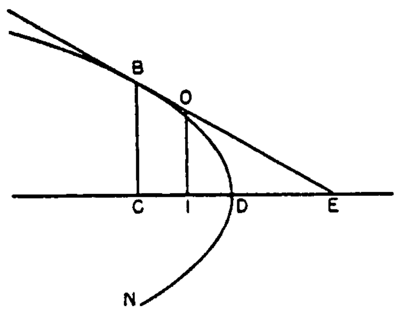

FIG. 2

it at which the line BE is to be drawn tangent to the parabola and intersecting the diameter at E.

We choose on the segment BE a point O at which we draw the ordinate OI; also we construct the ordinate BC of the point B. We have then: $CD/DI > BC^2/OI^2$, since the point O is exterior to the parabola. But $BC^2/OI^2 = CE^2/IE^2$, in view of the similarity of triangles. Hence $CD/DI > CE^2/IE^2$.

Now the point B is given, consequently the ordinate BC, consequently the point C, hence also CD. Let $CD = d$ be this given quantity. Put $CE = a$ and $CI = e$; we obtain

$$\frac{d}{d-e} > \frac{a^2}{a^2+e^2-2ae}.$$ [4]

Removing the fractions:

$$da^2 + de^2 - 2dae > da^2 - a^2e.$$

Let us then adequate, following the preceding method; by taking out the common terms we find:

$$de^2 - 2dae \sim -a^2e,$$

or, which is the same,

$$de^2 + a^2e \sim 2dae.$$

Let us divide all terms by e:

$$de + a^2 \sim 2da.$$

On taking out de, there remains $a^2 = 2da$, consequently $a = 2d$.

Thus we have proved that CE is the double of CD—which is the result.

This method never fails and could be extended to a number of beautiful problems; with its aid, we have found the centers of gravity of figures bounded by straight lines or curves, as well as those of solids, and a number of other results which we may treat elsewhere if we have time to do so.

I have previously discussed at length with M. de Roberval[5] the quadrature of areas bounded by curves and straight lines as well as the ratio that the solids which they generate have to the cones of the same base and the same height.

Now follows the second illustration of Fermat's "e-method," where Fermat's e = Newton's o = Leibniz' dx.[6]

CENTER OF GRAVITY OF PARABOLOID OF REVOLUTION, USING THE SAME METHOD[7]

Let $CBAV$ (Fig. 3) be a paraboloid of revolution, having for its axis IA and for its base a circle of diameter CIV. Let us find its center of gravity by using the same method which we applied for maxima and minima and for the tangents of curves; let us illustrate, with new examples and with new and brilliant applications of this method, how wrong those are who believe that it may fail.

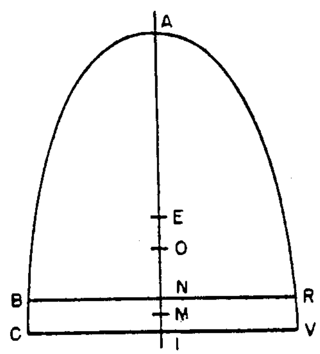

FIG. 3

In order to carry out this analysis, we write $IA = b$. Let O be the center of gravity, and a the unknown length of the segment AO; we intersect the axis IA by any plane BN and put $IN = e$, so that $NA = b - e$.

It is clear that in this figure and in similar ones (parabolas and paraboloids) the centers of gravity of segments cut off by parallels to the base divide the axis in a constant proportion (indeed, the argument of Archimedes can be extended by similar reasoning from the case of a parabola to all parabolas and paraboloids of revolution[8]). Then the center of gravity of the segment of which NA is the axis and BN the radius of the base will divide AN at a point E such that $NA/AE = IA/AO$, or, in formula, $b/a = (b - e)/AE$.

The portion of the axis will then be $AE = (ba - ae)/b$ and the interval between the two centers of gravity, $OE = ae/b$.

Let M be the center of gravity of the remaining part $CBRV$; it must necessarily fall between the points N, I, inside the figure, in view of Archimedes' postulate 9 in *On the equilibrium of planes*, since $CBRV$ is a figure completely concave in the same direction.[9]

But

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{OE}{OM},$$

since O is the center of gravity of the whole figure CAV and E and M are those of the parts.

Now in the paraboloid of Archimedes,

$$\frac{\text{Part } CAV}{\text{Part } BAR} = \frac{IA^2}{NA^2} = \frac{b^2}{b^2+e^2-2be};$$

hence by dividing,

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{2be-e^2}{b^2+e^2-2be}.$$

But we have proved that

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{OE}{OM}.$$

Then in formulas,

$$\frac{2be-e^2}{b^2+e^2-2be} = \frac{OE\ (= ae/b)}{OM};$$

hence

$$OM = \frac{b^2ae + ae^3 - 2bae^2}{2b^2a - be^2}.$$

From what has been established we see that the point M falls between points N and I; thus $OM < OI$; now, in formula, $OI = b - a$. The question is then prepared from our method, and we may write

$$b - a \sim \frac{b^2ae + ae^3 - 2bae^2}{2b^2e - be^2}.$$

Multiplying both sides by the denominator and dividing by e:

$$2b^3 - 2b^2a - b^2e + bae \sim b^2a + ae^2 - 2bae.$$

Since there are no common terms, let us take out those in which e occurs and let us equate the others:

$$2b^3 - 2b^2a = b^2a, \text{ hence } 3a = 2b.$$

Consequently

$$\frac{IA}{AO} = \frac{3}{2}, \text{ and } \frac{AO}{OI} = \frac{2}{1},$$

and this was to be proved.[10]

The same method applies to the centers of gravity of all the parabolas ad infinitum as well as those of paraboloids of revolution. I do not have time to indicate, for example, how to look for the center of gravity in our paraboloid obtained by revolution about the ordinate;[11] it will be sufficient to say that, in this conoid, the center of gravity divides the axis into two segments in the ratio 11/5.

NOTES

1. This paper was sent by Fermat to Father Marin Mersenne, who forwarded it to Descartes. Descartes received it in January 1638. It became the subject of a polemic discussion between him and Fermat (*Oeuvres*, I, 133). On Mersenne, see Selection I.6, note 1, of Struik.

2. See Selection IV.7, note 5, of Struik. where we have written (following the French translation in *Oeuvres*, III, 122) $be \sim 2ae + e^2$, Fermat wrote: *B* in *E* adaequabitur *A* in *E* bis + *Eq* (*Eq* standing for *E* quadratum). The symbol ~ is used for "adequates."

4. Fermat wrote: *D* ad *D* − *E* habebit majorem proportionem quam *Aq.* ad *Aq.* +

Eq. − *A* in *E* bis (*D* will have to *D* − *E* a larger ratio than A^2 to $A^2 + E^2 - 2AE$).

5. See the letters from Fermat to Roberval, written in 1636 (*Oeuvres*, III, 292-294, 296-297).

6. The gist of this method is that we change the variable x in $f(x)$ to $x + e$, e small. Since $f(x)$ is stationary near a maximum or minimum (Kepler's remark), $f(x + e) - f(x)$ goes to zero faster than e does. Hence, if we divide by e, we obtain an expression that yields the required values for x if we let e be zero. The legitimacy of this procedure remained, as we shall see, a subject of sharp controversy for many years. Now we see in it a first approach to the modern formula: $f'(x) = \lim_{e\to 0} \frac{f(x+e) - f(x)}{e}$, introduced by Cauchy (1820-21).

7. This paper seems to have been sent in a letter to Mersenne written in April 1638, for transmission to Roberval. Mersenne reported its contents to Descartes. Fermat used the term "parabolic conoid" for what we call "paraboloid of revolution."

8. "All parabolas" means "parabolas of higher order," $y = kx^n$, $n > 2$. The reference is to Archimedes' *On floating bodies*, II, Prop. 2 and following; see T. L. Heath, *The works of Archimedes* (Cambridge University Press, Cambridge, England, 1897; reprint, Dover, New York), 264ff.

9. This is postulate 7 in the modern Heiberg edition, and is translated in Heath, p. 190, as follows: "In any figure whose perimeter is concave in (one and) the same direction the center of gravity must be within the figure." (On the term "concave in the same direction," see Heath, p. 2.)

10. These relations were known to Archimedes (see note 8). But Fermat solved this problem on centers of gravity, hence a problem in the integral calculus, with what we might call an application of the principle of virtual variations.

11. Here *ACI* of Fig. 3 is rotated about *CI*.

71. From "On the Sines of a Quadrant of a Circle" (1659)*

(In the mid-17th century, French mathematician Gilles Roberval proposed the cycloid, or roulette as he called it, as a test curve for different methods relating to infinitesimals. The cycloid became the "apple of discord" among geometers. Roberval also introduced its so-called companion, that is, the sine curve. He influenced Pascal, who integrated $\sin^n x$, $n = 1, 2, 3, 4$, by means of an early form of a characteristic triangle, that is, dx, dy, and ds. Pascal's paper given here partially rejects indivisibles and presages the indefinite integral.)

BLAISE PASCAL

Let *ABC* [Fig. 1] be a quadrant of a circle of which the radius *AB* will be considered the axis and the perpendicular radius *AC* the base; let *D* be any point on the arc from which the sine *DI* will be drawn to the radius *AC*; and let *DE* be the tangent on which we choose the points *E* arbitrarily, and from these points we draw the perpendiculars *ER* to the radius *AC*.[1]

*Source: This translation of *Traité des sinus du quart de cercle* is made from the *Oeuvres de Pascal*, edited by L. Brunschwicg and P. Boutroux, and appears in D. J. Struik (ed.), *A Source Book in Mathematics, 1200-1800* (1969), 239-241. It is reprinted by permission of Harvard University Press, Copyright © 1969 by the President and Fellows of Harvard College. The biography of Blaise Pascal precedes selection 66.

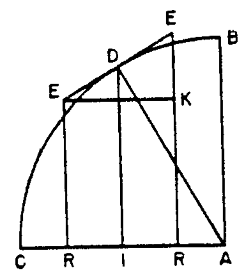

FIG. 1

I say that the rectangle formed by the sine[2] *DI* and the tangent *EE* is equal to the rectangle formed by a portion of the base (enclosed between the parallels) and the radius *AB*.

For the radius *AD* is to the sine *DI* as *EE* is to *RR*, or to *EK*, which is clear because of the similarity of the right-angled triangles *DIA*, *EKE*, the angle *EEK* or *EDI* being equal to the angle *DAI*.

Proposition I. The sum of the sines of any arc of a quadrant is equal to the portion of the base between the extreme sines, multiplied by the radius.[3]

Proposition II. The sum of the squares of those sines is equal to the sum of the ordinates[4] of the quadrant that lie between the extreme sines, multiplied by the radius.[5]

Proposition III. The sum of the cubes of the same sines is equal to the sum of the squares of the same ordinates between the extreme sines, multiplied by the radius.[6]

Proposition IV. The sum of the fourth powers of the same sines is equal to the sum of the cubes of the same ordinates between the extreme sines, multiplied by the radius.

And so on to infinity.

Preparation for the proof. Let any arc *BP* be divided into an infinite number of parts by the points *D* [Fig. 3] from which we draw the sines *PO*, *DI*, etc. . . . ; let us take in the other quadrant of the circle the segment *AQ*, equal to *AO* (which measures the distance between the extreme sines of the arc, *BA*, *PO*); let *AQ* be divided into an infinite number of equal parts by the points *H*, at which the ordinates *HL* will be drawn.

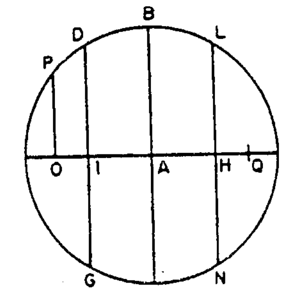

FIG. 3

Proof of Proposition I. I say that the sum of the sines *DI* (each of them multiplied of course by one of the equal small arcs *DD*) is equal to the segment *AO* multiplied by the radius *AB*.

Indeed, let us draw at all the points *D* the tangents *DE* [Fig. 1], each of which intersects its neighbor at the points *E*; if we drop the perpendiculars *ER* it is clear that each sine *DI* multiplied by the tangent *EE* is equal to each distance *RR* multiplied by the radius *AB*. Therefore, all the quadrilaterals formed by the sines *DI* and their tangents *EE* (which are all equal to each other) are equal to all the quadrilaterals formed by all the portions *RR* with the radius *AB*; that is (since one of the tangents *EE* multiplies each of the sines, and since the radius *AB* multiplies each of the distances), the sum of the sines *DI*, each of them multiplied by one of the tangents *EE*, is equal to the sum of the distances *RR*, each multiplied by *AB*. But each tangent *EE* is equal to each one of the equal arcs *DD*. Therefore the sum of the sines multiplied by one of the equal small arcs is equal to the distance *AO* [Fig. 3] multiplied by the radius.

Note. It should not cause surprise when I say that all the distances *RR* are equal to *AO* and likewise that each tan-