72. From "A New Method for Maxima and Minima as Well as Tangents, Which is Impeded Neither by Fractional nor by Irrational Quantities, and a Remarkable Type of Calculus for This" (1684)*

(Differential Calculus)

GOTTFRIED WILHELM LEIBNIZ

Let an axis AX [Fig. 1; simplified from Leibniz's figure] and several curves such as VV, WW, YY, ZZ be given, of which the ordinates VX, WX, YX, ZX, perpendicular to the axis, are called v, w, y, z respectively. The segment AX, cut off from the axis [*abscissa ab axe*[1]] is called x. Let the tangents be VB, WC, YD, ZE,

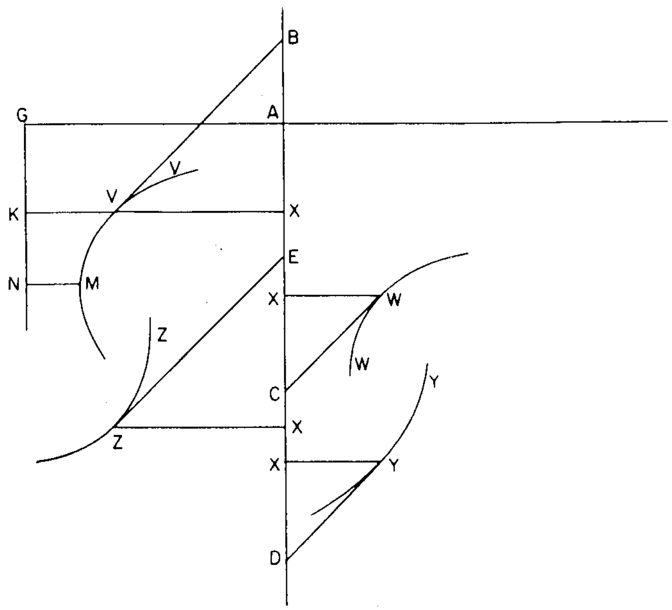

FIG. 1

**Source:* This translation of "Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus" is taken from D. J. Struik (ed.), *A Source Book in Mathematics, 1200-1800* (1969), 272-280, and is reprinted by permission of Harvard University Press, Copyright © 1969 by the President and Fellows of Harvard College.

intersecting the axis respectively at B, C, D, E. Now some straight line selected arbitrarily is called dx, and the line which is to dx as v (or w, or y, or z) is to XB (or XC, or XC, or XE) is called dv (or dw, or dy, or dz),[2] or the difference[3] of these v (or w, or y, or z). Under these assumptions we have the following rules of the calculus.

If a is a given constant, then $da = 0$, and $d(ax) = a\,dx$. If $y = v$ (that is, if the ordinate of any curve YY is equal to any corresponding ordinate of the curve VV), then $dy = dv$. Now *addition* and *subtraction:* if $z - y + w + x = v$, then $d(z - y + w + x) = dv = dz - dy + dw + dx$. Multiplication: $d(xv) = x\,dv + v\,dx$, or, setting $y = xv$, $dy = x\,dv + v\,dx$. It is indifferent whether we take a formula such as xv or its replacing letter such as y. It is to be noted that x and dx are treated in this calculus in the same way as y and dy, or any other indeterminate letter with its difference. It is also to be noted that we cannot always move backward from a differential equation without some caution, something which we shall discuss elsewhere.

Now *division:*

$$d\frac{v}{y} \text{ or } \left(\text{if } z = \frac{v}{y}\right) dz = \frac{\pm v\,dy \mp y\,dv}{yy}.$$

The following should be kept well in mind about the *signs.*[4] When in the calculus for a letter simply its differential is substituted, then the signs are preserved; for z we write dz, for $-z$ we write $-dz$, as appears from the previously given rule for addition and subtraction. However, when it comes to an explanation of the values, that is, when the relation of z to x is considered, then we can decide whether dz is a positive quantity or less than zero (or negative). When the latter occurs, then the tangent ZE is not directed toward A, but in the opposite direction, down from X. This happens when the ordinates z decrease with increasing x. And since the ordinates v sometimes increase and sometimes decrease, dv will sometimes be positive and sometimes be negative; in the first case the tangent VB is directed toward A, in the latter it is directed in the opposite sense. None of these cases happens in the intermediate position at M, at the moment when v neither increases nor decreases, but is stationary. Then $dv = 0$, and it does not matter whether the quantity is positive or negative, since $+0 = -0$. At this place v, that is, the ordinate LM, is *maximum* (or, when the convexity is turned to the axis, *minimum*), and the tangent to the curve at M is directed neither in the direction from X up to A, to approach the axis, nor down to the other side, but is parallel to the axis. When dv is infinite with respect to dx, then the tangent is perpendicular to the axis, that is, it is the ordinate itself. When $dv = dx$, then the tangent makes half a right angle with the axis. When with increasing ordinates v its increments or differences dv also increase (that is, when dv is positive, $d\,dv$, the difference of the differences, is also positive, and when dv is negative, $d\,dv$ is also negative), then the curve turns toward the axis its *concavity,* in the other case its *convexity.*[5] Where the increment is maximum or minimum, or where the increments from decreasing turn into increasing, or the opposite, there is a *point of inflection.*[6] Here concavity and convexity are interchanged, provided the ordinates too do not turn from increasing into decreasing or the opposite, because then the concavity or convexity would remain. However, it is impossible that the increments continue to increase or decrease, but the ordinates turn from increasing into decreasing, or the opposite.[7] Hence a point of inflection occurs when $d\,dv = 0$ while neither v nor $dv = 0$. The problem of finding inflection therefore has not, like that of finding a maximum, two equal roots, but three. This all depends on the correct use of the signs.

Sometimes it is better to use *ambiguous signs,* as we have done with the division, before it is determined what the precise sign is. When with increasing

$x\ v/y$ increases (or decreases), then the ambiguous signs in $d\frac{v}{y} = \frac{\pm v\,dy \mp y\,dv}{yy}$ must be determined in such a way that this fraction is a positive (or negative) quantity. But $\mp$ means the opposite of $\pm$, so that when one is $+$ the other is $-$ or vice versa. There also may be several ambiguities in the same computation, which I distinguish by parentheses. For example, let $\frac{v}{y} + \frac{y}{z} + \frac{x}{v} = w$; then we must write

$$\frac{\pm v\,dy \mp y\,dv}{yy} + \frac{(\pm)y\,dz\,(\mp)z\,dy}{zz} + \frac{((\pm))x\,dv\,((\mp))v\,dx}{vv} = dw,$$

so that the ambiguities in the different terms may not be confused. We must take notice that an ambiguous sign with itself gives $+$, with its opposite gives $-$, while with another ambiguous sign it forms a new ambiguity depending on both.

Powers. $dx^a = ax^{a-1}\,dx$; for example, $dx^3 = 3x^2\,dx$. $d\frac{1}{x^a} = -\frac{a\,dx}{x^{a+1}}$; for example, if $w = \frac{1}{x^3}$, then $dw = -\frac{3\,dx}{x^4}$.

Roots. $d\sqrt[b]{x^a} = \frac{a}{b}\,dx\sqrt[b]{x^{a-b}}$ (hence $d\sqrt[2]{y} = \frac{dy}{2\sqrt[2]{y}}$, for in this case $a = 1$, $b = 2$), therefore $\frac{a}{b}\sqrt[b]{x^{a-b}} = \frac{1}{2}\sqrt[2]{y^{-1}}$, but y^{-1} is the same as $\frac{1}{y}$; from the nature of the exponents in a geometric progression, and $\sqrt[2]{\frac{1}{y}} = \frac{1}{\sqrt[2]{y}}$, $d\frac{1}{\sqrt[b]{x^a}} = \frac{-a\,dx}{b\sqrt[b]{x^{a+b}}}$.

The law for integral powers would have been sufficient to cover the case of fractions as well as roots, for a power becomes a fraction when the exponent is negative, and changes into a root when the exponent is fractional. However, I prefer to draw these conclusions myself rather than relegate their deduction to others, since they are quite general and occur often. In a matter that is already complicated in itself it is preferable to facilitate the operations.

Knowing thus the *Algorithm* (as I may say) of this calculus, which I call *differential calculus*, all other differential equations can be solved by a common method. We can find maxima and minima as well as tangents without the necessity of removing fractions, irrationals, and other restrictions, as had to be done according to the methods that have been published hitherto. The demonstration of all this will be easy to one who is experienced in these matters and who considers the fact, until now not sufficiently explored, that *dx, dy, dv, dw, dz* can be taken proportional to the momentary differences, that is, increments or decrements, of the corresponding *x, y, v, w, z*. To any given equation we can thus write its differential equation. This can be done by simply substituting for each *term* (that is, any part which through addition or subtraction contributes to the equation) its differential quantity. For any other quantity (not itself a term, but contributing to the formation of the term) we use its differential quantity, to form the differential quantity of the term itself, not by simple substitution, but according to the prescribed Algorithm. The methods published before have no such transition. They mostly use a line such as *DX* or of similar kind, but not the line *dy* which is the fourth proportional to *DX, DY, dx*—something quite confusing. From there they go on removing fractions and irrationals (in which undetermined quantities occur). It is clear that our method also covers transcendental[8] curves—those that cannot be reduced by algebraic computation, or have no particular degree—and thus holds in a most general way without any particular and not always satisfied assumptions.

We have only to keep in mind that to find a *tangent* means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the

place of the *curve*. This infinitely small distance can always be expressed by a known differential like *dv*, or by a relation to it, that is, by some known tangent. In particular, if *y* were a transcendental quantity, for instance the ordinate of a cycloid, and it entered into a computation in which *z*, the ordinate of another curve, were determined, and if we desired to know *dz* or by means of *dz* the tangent of this latter curve, then we should by all means determine *dz* by means of *dy*, since we have the tangent of the cycloid. The tangent to the cycloid itself, if we assume that we do not yet have it, could be found in a similar way from the given property of the tangent to the circle.

Now I shall propose an example of the calculus, in which I shall indicate division by $x:y$, which means the same as x divided by y, or $\frac{x}{y}$.[9] Let the *first* or given equation be[10] $x:y + (a + bx)(c - xx):(ex + fxx)^2 + ax\sqrt{gg + yy} + yy:\sqrt{hh + lx + mxx} = 0$. It expresses the relation between x and y or between *AX* and *XY*, where *a, b, c, e, f, g, h* are given. We wish to draw from a point *Y* the line *YD* tangent to the curve, or to find the ratio of the line *DX* to the given line *XY*. We shall write for short $n = a + bx$, $p = c - xx$, $q = ex + fxx$, $r = gg + yy$, and $s = hh + lx + mxx$. We obtain $x:y + np:qq + ax\sqrt{r} + yy:\sqrt{s} = 0$, which we call the *second* equation. From our calculus it follows that

$$d(x:y) = (\pm x\,dy \mp y\,dx):yy,^{11}$$

and equally that

$$d(np:qq) = [(\pm)2np\,dq\,(\mp)q(n\,dp + p\,dn)]:q^3,$$

$$d(ax\sqrt{r}) = +ax\,dr:2\sqrt{r} + a\,dx\sqrt{r},$$

$$d(yy:\sqrt{s}) = ((\pm))yy\,ds\,((\mp))\,4ys\,dy:2s\sqrt{s}.$$

All these differential quantities from $d(x:y)$ to $d(yy:\sqrt{s})$ added together give 0, and thus produce a *third* equation, obtained from the terms of the second equation by substituting their differential quantities. Now $dn = b\,dx$ and $dp = -2x\,dx$, $d = e\,dx + 2fx\,dx$, $dr = 2y\,dy$, and $ds = l\,dx + 2mx\,dx$. When we substitute these values into the third equation we obtain a *fourth equation, in which the only remaining differential quantities, namely dx, dy,* are all outside of the denominators and without restrictions. Each term is multiplied either by *dx* or by *dy*, so that the law of homogeneity always holds with respect to these two quantities, however complicated the computation may be. From this we can always obtain the value of *dx:dy*, the ratio of *dx* to *dy*, or the ratio of the required *DX* to the given *XY*. In our case this ratio will be (if the fourth equation is changed into a proportionality):

$$\mp x:yy - axy:\sqrt{r}\,(\mp)\,2y:\sqrt{s}$$

divided by

$$\mp 1:y\,(\pm)\,(2npe + 2fx):q^3\,(\mp)\,(-2nx + pb):qq + a\sqrt{r}((\pm))yy(l + 2mx):2s\sqrt{s}.$$

Now *x* and *y* are given since point *Y* is given. Also given are the values of *n, p, q, r, s* expressed in *x* and *y*, which we wrote down above. Hence we have obtained what we required. Although this example is rather complicated we have presented it to show how the above mentioned rules can be used even in a more difficult computation. Now it remains to show their use in cases easier to grasp.

Let two points *C* and *E* [Fig. 2] be given and a line *SS* in the same plane. It is required to find a point *F* on *SS* such that when *E* and *C* are connected with *F* the sum of the rectangle of *CF* and a

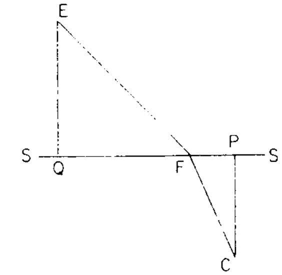

FIG. 2

given line h and the rectangle of FE and a given line r are as small as possible.[12] In other words, if SS is a line separating two media, and h represents the density of the medium on the side of C (say water), r that of the medium on the side of E (say air), then we ask for the point F such that the path from C to E via F is the shortest possible. Let us assume that all such possible sums of rectangles, or all possible paths, are represented by the ordinates KV of curve VV perpendicular to the line GK [Fig. 1]. We shall call these ordinates w. Then it is required to find their minimum NM. Since C and E [Fig. 2] are given, their perpendiculars to SS are also given, namely CP (which we call c) and EQ (which we call e); moreover PQ (which we call p) is given. We denote $QF = GN$ (or AX) by x, CF by f, and EF by g. Then $FP = p - x$, $f = \sqrt{cc + pp - 2px + xx}$ or $= \sqrt{l}$ for short; $g = \sqrt{ee + xx}$ or $= \sqrt{m}$ for short. Hence

$$w = h\sqrt{l} + r\sqrt{m}.$$

The differential equation (since $dw = 0$ in the case of a minimum) is, according to our calculus,

$$0 = +h\, dl : 2\sqrt{l} + r\, dm : 2\sqrt{m}.$$

But $dl = -2(p - x)\, dx$, $dm = 2x\, dx$; hence

$$h(p - x) : f = rx : g.$$

When we now apply this to dioptrics, and take f and g, that is, CF and EF, equal to each other (since the refraction at the point F is the same no matter how long the line CF may be), then $h(p - x) = rx$ or $h : r = x : (p - x)$, or $h : r = QF : FP$; hence the sines of the angles of incidence and of refraction, FP and QF, are in inverse ratio to r and h, the densities of the media in which the incidence and the refraction take place. However, this density is not to be understood with respect to us, but to the resistance which the light rays meet. Thus we have a demonstration of the computation exhibited elsewhere in these *Acta* [1682], where we presented a general foundation of optics, catoptrics, and dioptrics.[13] Other very learned men have sought in many devious ways what someone versed in this calculus can accomplish in these lines as by magic.

This I shall explain by still another example. Let *13* [Fig. 3] be a curve of such a nature that, if we draw from one of its points, such as *3*, six lines *34, 35, 36, 37, 38, 39* to six fixed points *4, 5, 6, 7, 8, 9* on the axis, then their sum is equal to a given line. Let *T14526789* be the axis, *12* the abscissa, *23* the ordinate, and let the tangent *3T* be required.

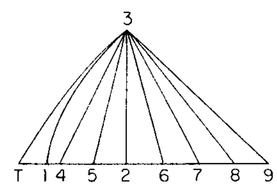

FIG. 3

Then I claim that $T2$ is to 23 as $\frac{23}{34} + \frac{23}{35} + \frac{23}{36} + \frac{23}{37} + \frac{23}{38} + \frac{23}{39}$ to $-\frac{24}{34} - \frac{25}{35} + \frac{26}{36} + \frac{27}{37} + \frac{28}{38} + \frac{29}{39}$. The same rule will hold if we increase the number of terms, taking not six but ten or more fixed points. If we wanted to solve this problem by the existing tangent methods, removing irrationals, then it would be a most tedious and sometimes insuperable task; in this case we would have to set up the condition that the rectangular planes and solids which can be constructed by means of all possible combinations of two or three of these lines are equal to a given quantity.[14] In all these cases and even in more complicated ones our methods are of astonishing and unequaled facility.

And this is only the beginning of much more sublime Geometry, pertaining to even the most difficult and most beautiful problems of applied mathematics, which without our differential calculus or something similar no one could attack with any such ease. We

shall add as appendix the solution of the problem which De Beaune proposed to Descartes and which he tried to solve in Vol. 3 of the *Letters*, but without success.[15] It is required to find a curve WW such that, its tangent WC being drawn to the axis, XC is always equal to a given constant line a. Then XW or w is to XC or a as dw is to dx. If dx (which can be chosen arbitrarily) is taken constant, hence always equal to, say, b, that is, x or AX increases uniformly, then $w = \frac{a}{b}\, dw$. Those ordinates w are therefore proportional to their dw, their increments or differences, and this means that if the x form an arithmetic progression, then the w form a geometric progression. In other words, if the w are numbers, the x will be logarithms, so that the curve WW is logarithmic.

NOTES

1. Note the Latin term *abscissa*. This term, which was not new in Leibniz's day, was made by him into a standard term, as were so many other technical terms. In the article "De linea ex lineis numero infinitis ordinatim ductis inter se concurrentibus formata . . . ," *Acta Eruditorum 11* (1692), 168-171 (Leibniz, *Mathematische Schriften*, Abth. 2, Band I (1858), 266-269), in which Leibniz discusses evolutes, he presents a collection of technical terms. Here we find *ordinata, evolutio, differentiare, parameter, differentiabilis, functio,* and *ordinata* and *abscissa* together designated as *coordinatae.* Here he also points out that ordinates may be given not only along straight but also along curved lines. The term *ordinate* is derived from *rectae ordinatim applicatae,* "straight lines designated in order," such as parallel lines. The term *functio* appears in the sentence: "the tangent and some other functions depending on it, such as perpendiculars from the axis conducted to the tangent."

2. When the subtangent—a term Leibniz used in a paper in the *Acta Eruditorum* (1694; *Mathematische Schriften,* Abth. 2, Band I, 306), though it may be older—is denoted by δ, Leibniz defines $dy : dx = y : \delta$, or $\delta = y : dy/dx$. We may express this by saying that Leibniz takes the derivative (geometrically, in the form of the tangent) without further definition, and defines the differentials in terms of the derivative.

3. Leibniz uses the term *differentia* and conceives it as a finite line segment. What we now call *differential* would long after Leibniz often be called *difference.* Leibniz also uses other terms. As to the meaning of the differentials, see the end of this selection.

4. The ambiguity in signs is due to the fact that s is taken positive. Systematic discrimination between positive and negative senses in analytic geometry came only with Monge and Möbius in the early nineteenth century.

5. Leibniz has "concavity" and "convexity" interchanged.

6. Leibniz' term is *punctum flexii contrarii* (point of opposite flection). On this term see T. F. Mulcrone, *The Mathematics Teacher 61* (1968), 475-478.

7. There seems to be something wrong here: when $y = x^2$, $dy = 2x\, dx$; then, when x passes from negative to positive ($dx > 0$), dy increases while y first decreases and then increases. However, see note 4.

8. This may be the first time that the term "transcendental" in the sense of "nonalgebraic" occurs in print.

9. From this suggestion by Leibniz dates the general adoption of this notation; see J. Tropfke, *Geschichte,* 3rd ed., II (1933), 30. See also the reference to Mengoli in G. Castelnuovo, *Le origini del calcolo infinitesimale nell'era moderna,* 153.

10. We have retained Leibniz' notation : but substituted parentheses for superscript bars: Leibniz writes

$$x : y + \overline{\overline{a + bx\; c\; xx}} : \text{quadrat. } ex + fxx + ax\sqrt{gg + yy} + yy : \sqrt{hh + lx + mxx} \text{ aequ. } 0.$$

11. Leibniz writes d, $x : y$.

12. For this problem, due to Fermat (*Oeuvres,* II (1844), 457), see note 13.

13. In this paper, "Unicum opticae, catoptricae et dioptricae principium," *Acta Eruditorum 1* (1683), 186-190, dealing with the laws of refraction and reflection, Leibniz makes known for the first time in print that he has his own *methodus de maximis et minimis.*

14. If the coordinates of point i are a_i, $i = 4, 5, \ldots$, and those of point 3 are x, y, then

this result can immediately be obtained by differentiating $\Sigma_i \sqrt{(x - a_i)^2 + y^2}$. Leibniz writes -24, -25 because his segments are all positive.

15. This is an inverse-tangent problem; Leibniz quotes *Les lettres de René Descartes* (3 vols.; Paris, ed. C. de Clerselier, 1657-1667). The problem is part of a long series of investigations that begins with the invention of logarithms by Napier by comparing an arithmetic and a geometric series and leads up to the full recognition of the inverse relation of the two functions $y = \log x$ and $x = e^y$ by Euler. Florimond De Beaune (1601-1652), a jurist at Blois, had written to Descartes about some curves; Descartes's answer of 1639 exists (Descartes, *Oeuvres*, II, 510-519); it was printed in the above-mentioned seventeenth-century edition of Descartes's letter and was studied by Leibniz. One of the curves was defined by a geometric description equivalent to the equation $dy/dx = (x - y)/b$. By means of the substitution $x' = b - x + y$, $y' = y$ this equation is transformed into $dy'/dx' = -y'/b$, the differential equation of what we now call the logarithmic curve. Descartes comes to the equivalent of this result; without mentioning logarithms he derives an inequality that can be written in our notation

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m-1} > \log\frac{m}{n} > \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m},$$

($n < m - 1$; m, n positive integers); see C. J. Scriba, "Zur Lösung des 2. Debeauneschen Problems durch Descartes," *Archive for History of Exact Sciences 1* (1961), 406-419. Descartes, like Napier, lets the logarithms grow when the argument decreases, while Briggs, who introduces 10 as base, lets argument and function grow at the same time. The next important steps, known to Leibniz, were Grégoire De Saint Vincent's determination of the area enclosed by a hyperbola, two ordinates, and an asymptote (1647), which Alfons Anton De Sarasa (1649) interpreted with the aid of logarithms, and Nikolaus Mercator's series (1667) for this area of the hyperbola: $a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \cdots$.

73. From "Supplementum geometriae dimensoriae . . ."[1] in *Acta Eruditorum* (1693)*

(The Fundamental Theorem of the Calculus)

GOTTFRIED WILHELM LEIBNIZ

I shall now show *that the general problem of quadratures can be reduced to the finding of a line that has a given law of tangency (declivitas)*, that is, for which the sides of the characteristic triangle have a given mutual relation. Then I shall show how this line can be described by a motion that I have invented. For this purpose [Fig. 1] I assume for every curve $C(C')$ a *double characteristic triangle*,[2] one, TBC, that is assignable, and one, GLC, that is inassignable,[3] and these two are similar. The inassignable triangle consists of the parts GL, LC, with the elements of the coordinates CF, CB as sides, and GC, the element of arc, as the base or hypotenuse. But the assignable triangle TBC consists of the axis, the ordinate, and the tangent, and therefore contains

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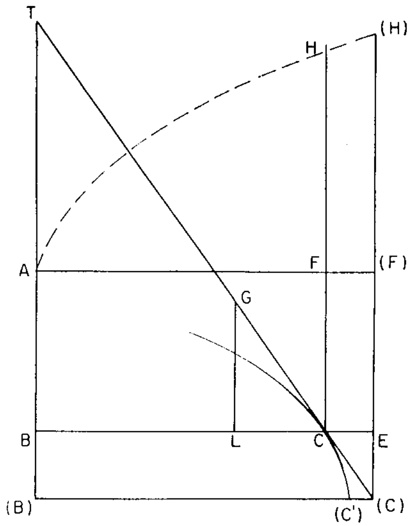

FIG. 1

the angle between the direction of the curve (or its tangent) and the axis or base, that is, the inclination of the curve at the given point C. Now let $F(H)$, the region of which the area has to be squared,[4] be enclosed between the curve $H(H)$, the parallel lines FH and $(F)(H)$, and the axis $F(F)$; on that axis let A be a fixed point, and let a line AB, the conjugate axis, be drawn through A perpendicular to AF. We assume that point C lies on HF (continued if necessary); this gives a new curve $C(C')$ with the property that, if from point C to the conjugate axis AB (continued if necessary) both its ordinate CB (equal to AF) and tangent CT are drawn, the part TB of the axis between them is to BC as HF to a constant [segment] a, or a times BT is equal to the rectangle AFH (circumscribed about the trilinear figure $AFHA$).[5] This being established, I claim that the rectangle on a and $E(C)$ (we must discriminate between the ordinates FC and $(F)(C)$ of the curve) is equal to the region $F(H)$. When therefore I continue line $H(H)$ to A, the trilinear figure $AFHA$ of the figure to be squared is equal to the rectangle with the constant a and the ordinate FC of the squaring curve as sides. This follows immediately from our calculus. Let $AF = y$, $FH = z$, $BT = t$, and $FC = x$; then $t = zy{:}a$, according to our assumption; on the other hand, $t = y\,dx{:}dy$ because of the property of the tangents expressed in our calculus. Hence $a\,dx = z\,dy$ and therefore $ax = \int z\,dy = AFHA$. Hence the curve $C(C')$ is the quadratrix with respect to the curve $H(H)$, while the ordinate FC of $C(C')$, multiplied by the constant a, makes the rectangle equal to the area, or the sum of the ordinates $H(H)$ corresponding to the corresponding abscissas AF. Therefore, since $BT : AF = FH : a$ (by assumption), and the relation of this FH to AF (which expresses the nature of the figure to be squared) is given, the relation of BT to FH or to BC, as well as that of BT to TC, will be given, that is, the relation between the sides of triangle TBC.[6] Hence, all that is needed to be able to perform the quadratures and measurements is to be able to describe the curve $C(C')$ (which, as we have shown, is the quadratrix), when the relation between the sides of the assignable characteristic triangle TBC (that is, the law of inclination of the curve) is given.

Leibniz continues by describing an instrument that can perform this construction.

NOTES

1. Leibniz distinguishes here between *geometria dimensoria*, which deals with quadratures and *geometria determmatrix*, which can be reduced to algebraic equations.

2. In Fig. 1 Leibniz assigns the symbol (C) to two points which we denote by (C) and (C'). If, with Leibniz, we write $CF = x$, $BC = y$, $HF = z$, then $E(C) = dx$, $CE = F(F) = dy$, and $H(H)(F)F = z\,dy$. First Leibniz introduces curve $C(C')$ with its characteristic triangle, and then later reintroduces it as the squaring curve [*curva quadratrix*] of curve $AH(H)$.

3. For want of anything better we use Leibniz's terms *assignabilis* and *inassign-*

abilis. G. Kowalewski, *Leibniz über die Analysis des Unendlichen*, Ostwald's *Klassiker*, No. 162 (Engelmann, Leipzig, 1908), 30, uses the German *angebbar* and *unangebbar*, "indicable" and "unindicable." For "differential" Leibniz in our text uses the term "element." Observe also the use of the term "coordinates" (Latin *coordinatae*).

4. The Latin is here a little more expressive than the English. From the Latin *quadrare* we can derive *quadrans, quadrandus, quadratrix, quadratura,* which can be translated by "to square," "squaring," "to be squared," "squaring curve" or "quadratrix," and "quadrature."

5. This is Pascal's expression; see Selections IV.11, 12 [in Struik].

6. This reasoning is still very much like that of Barrow, Gregory, and Torricelli, but because Leibniz possesses the converse relation $a\,dx = x\,dy$ — $\int a\,dx = \int x\,dy$ he needs only one demonstration, where Barrow needed two (Lecture X, 11; XI, 19; Selection IV.14 [in Struik]).

Chapter VI
The Scientific Revolution at Its Zenith (1620–1720)

Section C
The Discovery of the Differential and Integral Calculus

ISAAC NEWTON (1642–1727)

Isaac Newton was born on Christmas, the posthumous child of a yeoman father and Hannah (née Ayscough) Newton. At birth he was a physical weakling who, it was said, could have fit into a quart mug. He grew up in his father's house near the hamlet of Woolsthorpe. When his mother remarried, he was placed in the care of his maternal grandmother for eight years. The English Civil War had begun, and Cromwell was rising to power. Raids by armed men were frequent, and even in places where there was no immediate danger people lived in fear. Isaac, a solitary child without playmates, turned to meditation. He began to construct mechanical contrivances—fiery kites, lanterns, and models of mills. When her second husband died, his mother returned to Woolsthorpe in 1653 intending to make a farmer of Isaac. However, his uncle and the master of Grantham school convinced her that the boy was unsuited for such work and should be sent to a university.

In 1661, Newton was admitted to Trinity College, Cambridge, where he received the Bachelor of Arts degree in 1665, when the great plague was just beginning. As the disease spread across the country, universities were closed. From June 1665 through 1666, he stayed at home in Woolsthorpe but made at least one visit to Cambridge. In his musings as an elderly man, Newton remembered this period as an *annus mirabilis* during which he laid the foundations for his monumental scientific achievement. In mathematics, he discovered by induction the general binomial theorem and invented an early stage of the calculus. In optics, he decomposed white light with his prism, and, in mechanics, he enunciated the inverse-square law of attraction. His extraordinary scientific creativity was underway. His studies of the calculus and spectroscopy were in embryo, while in mechanics he continued to examine the Cartesian vortex theory, which he later rejected accepting attraction instead. After returning to Trinity College in 1667, he completed the M.A. degree in 1668, whereupon he was appointed Lucasian Professor of Mathematics, a position he held until 1701. Presumably, the first incumbent, Isaac Barrow, recognized Newton as a prodigy and resigned so that he might have this prestigious chair. Few students attended Newton's lectures in algebra and dynamics, fewer still understood them. His teaching departed from the dominant Cartesian physics presented in the school texts. He was a quirky, retiring scholar who was known to eat sparingly, skip meals,