74. From *Specimens of a Universal* [*System of*] *Mathematics* (written c. 1684)*

ISAAC NEWTON

A certain method of resolving problems by convergent series devised by me about eighteen years ago[1] had, by my very honest friend[2] Mr. John Collins, around that time been announced to Mr. James Gregory, the renowned Professor of Mathematics at Edinburgh University in Scotland, as being in my possession. This method Mr. Gregory learnt—a measure of the power of his intellect—from but a single series of this type which had been passed on to him, but not much aliorum afterwards he was snatched away by an untimely death.[3] From his papers, extant in which were certain calculations though without a description of the method, his celebrated successor in the Mathematical Chair, David Gregory, also learnt this method of calculation and developed it in a neat and stimulating tract: in this he revealed not only his expertise in mathematical topics but also the utmost probity, candidly acknowledging what he himself had taken from his predecessor and what his predecessor had received from Collins.[4] While still reading it[5] I began at once to reflect that, since I had often been asked to publish something, I had now less excuse to resist the entreaties of my friends and thwart the expectation of others,[6] and that I were better advised to be swiftly acquiescent rather than have to submit with annoyance at a later, less opportune time. Through the agency[7] of Mr. H. O. certain letters regarding series of this sort once passed between me and Mr. Leibniz, now the Duke of Hanover's eminent minister in public affairs. By publishing these I shall certainly oblige the reader more than if (after what Mr. Gregory has done) I should write up the whole subject afresh, and especially so since in them is contained Leibniz' extremely elegant method, far different from mine, of attaining the same series—one about which it would be dishonest to remain silent while publishing my own. Mine shall, indeed, be described in such a way that Mr. Gregory's book, with its more lavish explanation of points which are here touched upon piecemeal and omission of others which are here more copiously described, may profitably fill the rôle of an introduction. The letters, however, are to this effect.[8]

NOTES

1. That is, in 1665-6. Newton here seems to refer to his discovery of the general binomial expansion early in 1665 and perhaps the "mechanical" extraction of the root of an algebraic equation as an infinite series according to "Vieta's Analyticall resolution of powers" adumbrated in his October 1666 tract but not, of course, developed at length till he came to write his 1671 treatise.

2. This replaces the more fulsome phrase "rerum Mathematicarum cultor eximius" (that outstanding cultivator of things mathematical). Newton knew very well the moderate limits of Collins' mathematical talents and not even the latter's recent death (in November 1683) would provoke him more than momentarily to an undeserved epithet.

3. James Gregory died in late October 1675, when he was not quite 37 years old and still in the prime of life, within a few hours of having had a severe stroke accompanied by blindness and paralysis. Newton's source of information is David Gregory who wrote in his *Exercitatio Geometrica*. . . .

4. Compare David Gregory's *Exercitatio Geometrica*: 4. An overwhelming number of the examples to which David applies his rediscovered Gregorian method in the body of his book come in fact (though this is not always made clear in the case of still unpublished *adversaria*) from the papers and printed works of his uncle James, though he at one point refers to Nicolaus Mercator's *Logarithmotechnia* for the series expansion of log (1+x) and twice cites René-François de Sluse's *Miscellanea* (Liège, 1668), once as source for the general Slusian "pearl," once as inventor of the Slusian conchoid.

5. The subjunctive mood is manifestly a slip on Newton's part: the act of reading is coterminous with the reaction it inspired.

6. In the present mathematical context Newton would probably rank Barrow and Collins (now both dead) as friends, Oldenburg (also deceased seven years before) and John Wallis as "other" acquaintances!

7. "mediante" (through the mediation) is cancelled. "H.O." is, of course, "H[enrico] O[ldenburgo]" (Henry Oldenburg). The designation of Leibniz—after the spring of 1676 Librarian (and unofficial legal adviser) to the Duke of Hanover—as his "minister in negotijs publicis" (minister of state?) probably points to Newton's lack of awareness of contemporary political realities at this period.

8. In sequel Newton indicates extracts relating to infinite series from the five letters which passed (by way of Oldenburg) between himself and Leibniz during the period June 1676 to July 1677. We relate these brief citations of terminal phrases (quoted by Newton, in the case of the three Leibniz letters, from copies furnished him by Oldenburg and Collins) to the reproductions of the originals as sent, given by H. W. Turnbull in his edition of *The Correspondence of Isaac Newton*, 2, 1960. While rightly choosing not to tamper with the text of his *epistola prior* of 13 June 1676—nor, of course, with Leibniz' words—Newton has slightly revised and clarified passages in his *epistola posterior* of the following 24 October.

*Source: This translation of *Matheseos Universalis Specimena* is from D. T. Whiteside (ed.), *The Mathematical Papers of Isaac Newton*, vol. IV, 1674-1684 (1971), 527-531. It is reprinted by permission of Cambridge University Press. Footnotes are renumbered.

75. From a Letter to Henry Oldenburg on the Binomial Series (June 13, 1676)*

ISAAC NEWTON

Though the modesty of Mr. Leibniz, in the extracts from his letter which you have lately sent me, pays great tribute to our countrymen for a certain theory of infinite series, about which there now begins to be some talk, yet I have no doubt that he has discovered not only a method for reducing any quantities whatever to such series, as he asserts, but also various shortened forms, perhaps like our own, if not even better. Since, however, he very much wants to know what has been discovered in this subject by the English, and since I myself fell upon this theory some years ago, I have sent you some of those things which occurred to me in order to satisfy his wishes, at any rate in part.

Fractions are reduced to infinite series by division; and radical quantities by

*Source: This translation from H. W. Turnbull, F.R.S. (ed.), *The Correspondence of Isaac Newton*, vol. II (1960), 32-33. Reprinted by permission of Cambridge University Press. Henry Oldenburg was secretary of the Royal Society.

extraction of the roots, by carrying out those operations in the symbols just as they are commonly carried out in decimal numbers. These are the foundations of these reductions: but extractions of roots are shortened by this theorem,

$$(P + PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \text{etc.}$$

where $P + PQ$ signifies the quantity whose root or even any power, or the root of a power, is to be found; P signifies the first term of that quantity, Q the remaining terms divided by the first, and m/n the numerical index of the power of $P + PQ$, whether that power is integral or (so to speak) fractional, whether positive or negative. For as analysts, instead of aa, aaa, etc., are accustomed to write a^2, a^3, etc., so instead of $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c{:}a^5}$, etc. I write $a^{1/2}$, $a^{3/2}$, $a^{5/3}$, and instead of $1/a$, $1/aa$, $1/a^3$, I write a^{-1}, a^{-2}, a^{-3}. And so for

$$\frac{aa}{\sqrt{c:(a^3 + bbx)}}$$

I write $aa(a^3 + bbx)^{-1/3}$, and for

$$\frac{aab}{\sqrt{c:\{(a^3 + bbx)(a^3 + bbx)\}}}$$

I write $aab(a^3 + bbx)^{-2/3}$: in which last case, if $(a^3 + bbx)^{-2/3}$ is supposed to be $(P + PQ)^{m/n}$ in the Rule, then P will be equal to a^3, Q to bbx/a^3, m to -2, and n to 3. Finally, for the terms found in the quotient in the course of the working I employ A, B, C, D, etc., namely, A for the first term, $P^{m/n}$; B for the second term, $m/n\ AQ$; and so on. For the rest, the use of the rule will appear from the examples.

Example 1

$$\sqrt{(c^2+x^2)} \text{ or } (c^2+x^2)^{1/2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256|c|^9} + \text{etc.}$$

For in this case, $P = c^2$, $Q = x^2/c^2$, $m = 1$, $n = 2$,

$$A(= P^{m/n} = (cc)^{1/2}) = c,\ B(= (m/n)AQ) = x^2/2c,$$

$$C\left(= \frac{m-n}{2n}BQ\right) = -\frac{x^4}{8c^3};$$

and so on.

76. From Letter to Henry Oldenburg on General Method for Finding Quadratures[1] (October 24, 1676)*

ISAAC NEWTON

I can hardly tell with what pleasure I have read the letters of those very distinguished men Leibniz and Tschirnhaus. Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else. But what he has scattered elsewhere throughout his letter is most worthy of his reputation—it leads us also to hope for very great things from him. The variety of ways by which the same goal is approached has given me the greater pleasure, because three methods of arriving at series of that kind had already become known to me, so that I could scarcely expect a new one to be communicated to us. One of mine I have described before; I now add another, namely, that by which I first chanced on these series—for I chanced on them before I knew the divisions and extractions of roots which I now use. And an explanation of this will serve to lay bare, what Leibniz desires from me, the basis of the theorem set forth near the beginning of the former letter.

At the beginning of my mathematical studies, when I had met with the works of our celebrated Wallis, on considering the series by the intercalation of which he himself exhibits the area of the circle and the hyperbola, the fact that, in the series of curves whose common base or axis is x and the ordinates

$$(1-x^2)^{0/2},\ (1-x^2)^{1/2},\ (1-x^2)^{2/2},$$

$$(1-x^2)^{3/2},\ (1-x^2)^{4/2},\ (1-x^2)^{5/2},\ \text{etc.},$$

if the areas of every other of them, namely

$$x,\ x - \tfrac{1}{3}x^3,\ x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5,$$

$$x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7,\ \text{etc.}$$

could be interpolated, we should have the areas of the intermediate ones, of which the first $(1-x^2)^{1/2}$ is the circle: in order to interpolate these series I noted that in all of them the first term was x and that the second terms $\tfrac{0}{3}x^3$, $\tfrac{1}{3}x^3$, $\tfrac{2}{3}x^3$, $\tfrac{3}{3}x^3$, etc., were in arithmetical progression, and hence that the first two terms of the series to be intercalated ought to be $x - \tfrac{1}{3}(\tfrac{1}{2}x^3)$, $x - \tfrac{1}{3}(\tfrac{3}{2}x^3)$, $x - \tfrac{1}{3}(\tfrac{5}{2}x^3)$, etc. To intercalate the rest I began to reflect that the denominators 1, 3, 5, 7, etc. were in arithmetical progression, so that the numerical coefficients of the numerators only were still in need of investigation. But in the alternately given areas these were the figures of powers of the number 11, namely of these, 11^0, 11^1, 11^2, 11^3, 11^4, that is, first 1; then 1, 1; thirdly, 1, 2, 1; fourthly 1, 3, 3, 1; fifthly 1, 4, 6, 4, 1, etc. And so I began to inquire how the remaining figures in these series could

*Source: This translation from H. W. Turnbull, F.R.S. (ed.), *The Correspondence of Isaac Newton*, vol. II (1960), 130–134 and 148–149. Reprinted by permission of Cambridge University Press.

be derived from the first two given figures, and I found that on putting m for the second figure, the rest would be produced by continual multiplication of the terms of this series,

$$\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$$

$$\times\frac{m-3}{4}\times\frac{m-4}{5},\ \text{etc.}$$

For example, let $m=4$, and $4\times\frac{1}{2}(m-1)$, that is 6 will be the third term, and $6\times\frac{1}{3}(m-2)$, that is 4 the fourth, and $4\times\frac{1}{4}(m-3)$, that is 1 the fifth, and $1\times\frac{1}{5}(m-4)$, that is 0 the sixth, at which term in this case the series stops. Accordingly, I applied this rule for interposing series among series, and since, for the circle, the second term was $\frac{1}{3}(\frac{1}{2}x^3)$, I put $m=\frac{1}{2}$, and the terms arising were

$$\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}\ \text{or}\ -\frac{1}{8},$$

$$-\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}\ \text{or}\ +\frac{1}{16},$$

$$\frac{1}{16}\times\frac{\frac{1}{2}-3}{4}\ \text{or}\ -\frac{5}{128},$$

and so to infinity. Whence I came to understand that the area of the circular segment which I wanted was

$$x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{8}x^5}{5}-\frac{\frac{1}{16}x^7}{7}-\frac{\frac{5}{128}x^9}{9}\ \text{etc.}$$

And by the same reasoning the areas of the remaining curves, which were to be inserted, were likewise obtained: as also the area of the hyperbola and of the other alternate curves in this series $(1+x^2)^{0/2}$, $(1+x^2)^{1/2}$, $(1+x^2)^{2/2}$, $(1+x^2)^{3/2}$, etc. And the same theory serves to intercalate other series, and that through intervals of two or more terms when they are absent at the same time. This was my first entry upon these studies, and it had certainly escaped my memory, had I not a few weeks ago cast my eye back on some notes.

But when I had learnt this, I immediately began to consider that the terms

$$(1-x^2)^{0/2},\quad (1-x^2)^{2/2},$$

$$(1-x^2)^{4/2},\quad (1-x^2)^{6/2},\quad \text{etc.,}$$

that is to say,

$$1,\quad 1-x^2,\quad 1-2x^2+x^4,$$

$$1-3x^2+3x^4-x^6,\quad \text{etc.}$$

could be interpolated in the same way as the areas generated by them: and that nothing else was required for this purpose but to omit the denominators 1, 3, 5, 7, etc., which are in the terms expressing the areas; this means that the coefficients of the terms of the quantity to be intercalated $(1-x^2)^{1/2}$, or $(1-x^2)^{3/2}$, or in general $(1-x^2)^m$, arise by the continued multiplication of the terms of this series

$$m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4},\quad \text{etc.,}$$

so that (for example)

$(1-x^2)^{1/2}$ was the value of
$$1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}x^6\quad \text{etc.,}$$

$$(1-x^2)^{3/2}\ \text{of}\ 1-\tfrac{3}{2}x^2+\tfrac{3}{8}x^4+\tfrac{1}{16}x^6,\quad \text{etc.,}$$

and

$$(1-x^2)^{1/3}\ \text{of}\ 1-\tfrac{1}{3}x^2-\tfrac{1}{9}x^4-\tfrac{5}{81}x^6,\quad \text{etc.}$$

So then the general reduction of radicals into infinite series by that rule, which I laid down at the beginning of my earlier letter became known to me, and that before I was acquainted with the extraction of roots. But once this was known, that other could not long remain hidden from me. For in order to test these processes, I multiplied

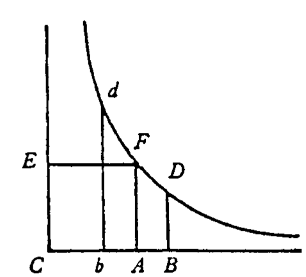
$$1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}x^6,\quad \text{etc.}$$

into itself; and it became $1-x^2$, the remaining terms vanishing by the continuation of the series to infinity. And even so $1-\frac{1}{3}x^2-\frac{1}{9}x^4-\frac{5}{81}x^6$, etc. multiplied twice into itself also produced $1-x^2$. And as this was not only sure proof of these conclusions so too it guided me to try whether, conversely, these series, which it thus affirmed to be roots of the quantity $1-x^2$, might not be extracted out of it in an arithmetical manner. And the matter turned out well. This was the form of the working in square roots.

$$1-x^2(1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}x^6,\ \text{etc.}$$
$$\underline{1}$$
$$0-x^2$$
$$\underline{-x^2+\tfrac{1}{4}x^4}$$
$$-\tfrac{1}{4}x^4$$
$$\underline{-\tfrac{1}{4}x^4+\tfrac{1}{8}x^6+\tfrac{1}{64}x^8}$$
$$0\quad -\tfrac{1}{8}x^6-\tfrac{1}{64}x^8.$$

After getting this clear I have quite given up the interpolation of series, and have made use of these operations only, as giving more natural foundations. Nor was there any secret about reduction by division, an easier affair in any case. But soon I attacked the resolution of affected equations and obtained it. Whence the ordinates, the segments of the axes and any other right lines at once became known from the areas or arcs of the curves being given. For the return to them needed nothing beyond the solution of the equations by which the areas or arcs were given in terms of the given right lines.

At that time the plague breaking out forced me to flee hence and think about other things. Yet, soon after, I added a certain way of finding logarithms from the area of an hyperbola, which I here append. Let dFD be an hyperbola, C its centre, F its vertex, and let $CAFE = 1$ be an inscribed square. In CA take AB, Ab, on this side and that, equal to $\frac{1}{10}$ or $0\cdot1$. Then, the perpendiculars BD, bd being erected to terminate on the



hyperbola, the semi-sum of the areas AD and Ad will be

$$=0\cdot1+\frac{0\cdot001}{3}+\frac{0\cdot00001}{5}+\frac{0\cdot0000001}{7}$$

and the semi-difference

$$=\frac{0\cdot01}{2}+\frac{0\cdot0001}{4}+\frac{0\cdot000001}{6}+\frac{0\cdot00000001}{8},\ \text{etc.}$$

which give on reduction

0·1000000000000	0·0050000000000
3333333333	250000000
20000000	1666666
142857	12500
1111	100
9	1
0·1003353477310	0·0050251679267

The sum of these, 0·1053605156577, is Ad, and the difference, 0·0953101798043, is AD. And in the same way, if AB, Ab are taken on this side and that, equal to 0·2, the result Ad = 0·2231435513142 and AD = 0·1823215567939 will be had. Thus, having obtained the hyperbolic logarithms of the four decimal numbers 0·8, 0·9, 1·1 and 1·2, since $(1\cdot2/0\cdot8)\times(1\cdot2/0\cdot9)=2$, and 0·8 and 0·9 are less than unity, add their logarithms to twice the logarithm of 1·2 and you will have 0·6931471805597 for the hyperbolic logarithm of the number 2. To the triple of this add log 0·8 (since $(2\times2\times2)/0\cdot8=10$) and you will have 2·3025850929933 for the logarithm of the number 10. Thence, by addition the logarithms of the numbers 9 and 11 follow at once; so that the logarithms of all the primes 2, 3, 5, 11 are in readiness. In addition, merely by lowering the numbers in the above calculation by decimal places, and by addition, the logarithms of the decimals 0·98, 0·99, 1·01, 1·02 are obtained; as also of 0·998, 0·999, 1·001, 1·002. And then by addition and subtraction the logarithms of the primes 7, 13, 17, 37, etc., emerge. And these, combined with the above and divided by the logarithm of the number 10, become true logarithms for inserting in the table. But afterwards I have obtained them more closely.

I am ashamed to tell to how many places I carried these computations, having no other business at that time: for then I took really too much delight in these inventions. But when there appeared that ingenious work, the *Logarithmotechnia* of Nicolas Mercator (whom I suppose to have made his discoveries first), I began to pay less attention to these things, suspecting that either he knew the extraction of roots as well as division of fractions, or at least that others upon the discovery of division would find out the rest before I could reach a ripe age for writing. Yet at the very time when this book appeared, a certain compendium of the method of these series was communicated by Mr. Barrow (then professor of mathematics) to Mr. Collins; in which I had indicated the areas and lengths of all curves, and the surfaces and volumes of solids from given right lines, and that conversely from these as given the right lines could be determined; and the method there disclosed I had illustrated by various series. When afterwards a regular correspondence developed between us, Collins, a man born to promote the art of mathematics, did not cease to suggest that I should make these things public. And five years ago when, urged by my friends, I had planned to publish a treatise on the refraction of light and on colours, which I then had in readiness, I began again to think about these series and I compiled a treatise on them too, with a view to publishing both at the same time. But on the occasion of the Reflecting Telescope, when I had sent you a letter in which I briefly explained my ideas of the nature of light, something unexpected caused me to feel that it was my business to write to you in haste about the printing of that letter. Then frequent interruptions that immediately arose from the letters of various persons (full of objections and of other matters) quite deterred me from the design and caused me to accuse myself of imprudence, because, in hunting for a shadow hitherto, I had sacrificed my peace, a matter of real substance.

About that time, from just a single one of my series which Collins had sent him, Gregory, after much reflection, as he wrote back to Collins, arrived at the same method, and he left a treatise on it which we hope is going to be published by his friends. Indeed, with his strong understanding he could not fail to add many discoveries of his own, and it is in the interest of mathematics that these should not be lost. Moreover, I myself had not completely finished my treatise when I desisted from the proposal, nor has my mind to this day returned to the task of adding the rest. In fact there was wanting that part in which I had decided to explain the mode of solving problems which cannot be reduced to squarings; although I had done something to lay its foundations.

But in that treatise infinite series played no great part. Not a few other things I brought together, among them the method of drawing tangents which the very skillful Sluse communicated to you two or three years ago, about which you wrote back [to him] (on the suggestion of Collins) that the same method had been known to me also. We happened on it by different reasoning: for, as I work it, the matter needs no proof. Nobody, if he possessed my basis, could draw tangents any other way, unless he were deliberately wandering from the straight path. Indeed we do not here stick at equations in radicals involving one or each indefinite quantity, however complicated they may be; but without any reduction of such equations (which would generally render the work endless) the tangent is drawn directly. And the same is true in questions of maxima and minima, and in some others too, of which I am not now speaking. The foundation of these operations is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus: 6*accdæ* 13*eff*7*i*3*l*9*n*4*o*4*qrr*4*s*8*t*12*vx*.

[*Portion of letter omitted.*]

THE ANAGRAM

This inverse problem of tangents, when the tangent between the point of contact and the axis of the figure is of given length, does not demand these methods. Yet it is that mechanical curve the determination of which depends on the area of an hyperbola. The problem is also of the same kind, when the part of the axis between the tangent and the ordinate is given in length. But I should scarcely have reckoned these cases among the sports of nature. For when in the right-angled triangle, which is formed by that part of the axis, the tangent and the ordinate, the relation of any two sides is defined by any equation, the problem can be solved apart from my general method. But when a part of the axis ending at some point given in position enters the bracket, then the question is apt to work out differently.

The communication of the solution of affected equations by the method of Leibniz will be very agreeable; so too an explanation how he comports himself when the indices of the powers are fractional, as in this equation.

$$20 + x^{3/7} - x^{6/5}y^{2/3} - y^{7/11} = 0,$$

or surds, as in

$$(x^{\sqrt{2}} + x^{\sqrt{7}})^{\sqrt[3]{2/3}} = y,$$

where $\sqrt{2}$ and $\sqrt{7}$ do not mean coefficients of x, but indices of powers or dignities of it, and $\sqrt[3]{2/3}$ means the power of the binomial $x^{\sqrt{2}} + x^{\sqrt{7}}$. The point, I think, is clear by my method, otherwise I should have described it. But a term must at last be set to this wordy letter. The letter of the most excellent Leibniz fully deserved of course that I should give it this more extended reply. And this time I wanted to write in greater detail because I did not believe that your more engaging pursuits should often be interrupted by me with this rather austere kind of writing.

1. *Turnbull's Note.* Oldenburg transmitted Newton's letter of June 13, 1676 to Leibniz, who responded in a letter (August 17, 1676), revealing his results in finding quadratures and hinting that he had a general method. Leibniz's letter interested Newton, who wrote a second letter to Oldenburg dated October 24, 1676. His second letter guardedly presented by means of an anagram a general method for finding quadratures and its inverse problem of tangents.