78. From the Introduction to the *Tractatus de quadratura curvarum* (1704)*

ISAAC NEWTON

1. I consider mathematical quantities in this place not as consisting of very small parts; but as described by a continued motion. Lines are described, and thereby generated not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by a continual flux: and so in other quantities. These geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles.

2. Therefore considering that quantities, which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated; I sought a method of determining quantities from the velocities of the motions or increments, with which they are generated; and calling these velocities of the motions or increments *fluxions*, and the generated quantities *fluents*. I fell by degrees upon the method of fluxions, which I have made use of here in the quadrature of curves, in the years 1665 and 1666.

3. Fluxions are very nearly as the augments of the fluents generated in equal but very small particles of time, and, to speak accurately, they are in the *first ratio* of the nascent augments; but they may be expounded by any lines which are proportional to them.

4. Thus if the areas *ABC*, *ABDG* [Fig. 1] be described by the ordinates *BC*, *BD* moving along the base *AB* with an uniform motion, the fluxions of these areas shall be to one another as the describing ordinates *BC* and *BD*, and may be expounded by these ordinates, because that these ordinates are as the nascent augments of the areas.

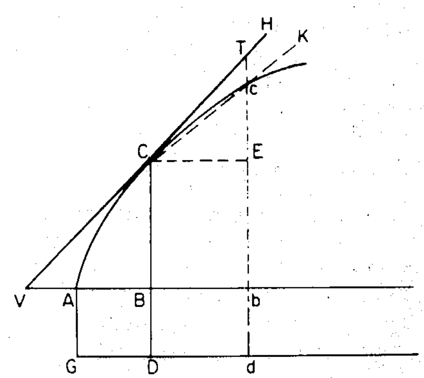

Fig. 1

5. Let the ordinate *BC* advance from its place into any new place *bc*. Complete the parallelogram *BCEb*, and draw the right line *VTH* touching the curve in *C*, and meeting the two lines *bc* and *BA* produced in *T* and *V*: and *Bb*, *Ec*, and *Cc* will be the augments now generated of the abscissa *AB*, the ordinate *BC* and the curve line *ACc*; and the sides of the triangle *CET* are in the *first ratio* of these augments considered as nascent, therefore the fluxions of *AB*, *BC*, and *AC* are as the sides *CE*, *ET*, and *CT* of that triangle *CET*, and may be expounded by these same sides, or, which is the same thing, by the sides of the triangle *VBC*, which is similar to the triangle *CET*.

6. It comes to the same purpose to take the fluxions in the *ultimate ratio* of the evanescent parts. Draw the right line *Cc*, and produce it to *K*. Let the ordinate *bc* return into its former place *BC*, and when the points *C* and *c* coalesce, the right line *CK* will coincide with the tangent *CH*, and the evanescent triangle *CEc* in its ultimate form will become similar to the triangle *CET*, and its evanescent sides *CE*, *Ec*, and *Cc* will be *ultimately* among themselves as the sides *CE*, *ET*, and *CT* of the other triangle *CET* are, and therefore the fluxions of the lines *AB*, *BC*, and *AC* are in this same ratio. If the points *C* and *c* are distant from one another by any small distance, the right line *CK* will likewise be distant from the tangent *CH* by a small distance. That the right line *CK* may coincide with the tangent *CH*, and the ultimate ratios of the lines *CE*, *Ec*, and *Cc* may be found, the points *C* and *c* ought to coalesce and exactly coincide. The very smallest errors in mathematical matters are not to be neglected.

7. By the like way of reasoning, if a circle described with the center *B* and radius *BC* be drawn at right angles along the absciss *AB*, with an uniform motion, the fluxion of the generated solid *ABC* will be as that generating circle, and the fluxion of its superficies will be as the perimeter of that circle and the fluxion of the curve line *AC* jointly. For in whatever time the solid *ABC* is generated by drawing that circle along the length of the absciss, in the same time its superficies is generated by drawing the perimeter of that circle along the length of the curve *AC*. You may likewise take the following examples of this method.

8. *Let the right line PB* [Fig. 2], *revolving about the given pole P, cut another right line AB given in position: it is required to find the proportion of the fluxions of these right lines AB and PB.*

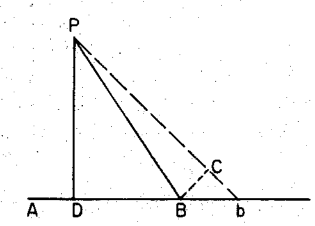

Fig. 2

Let the line *PB* move forward from its place *PB* into the new place *Pb*. In *Pb* take *PC* equal to *PB*, and draw *PD* to *AB* in such manner that the angle *bPD* may be equal to the angle *bBC*; and because the triangles *bBC*, *bPD* are similar, the augment *Bb* will be to the augment *Cb* as *Pb* to *Db*. Now let *Pb* return into its former place *PB*, that these augments may evanish, then the ultimate ratio of these evanescent augments, that is the ultimate ratio of *Pb* to *Db*, shall be the same with that of *PB* to *DB*, *PDB* being then a right angle, and therefore the fluxion of *AB* is to the fluxion of *PB* in that same ratio.

9. *Let the right line PB, revolving about the given pole P, cut other two right lines given in position, viz. AB and AE in B and E: the proportion of the fluxions of these right lines AB and AE is sought.*

Let the revolving right line *PB* [Fig. 3] move forward from its place *PB* into the new place *Pb*, so as to cut the lines *AB*, *AE* in the points *b* and e: and draw *BC* parallel to *AE* meeting *Pb* in *C*, and it will be *Bb*:*BC*::*Ab*:*Ae*, and *BC*:*Ee*::*PB*:*PE*, and by joining the ratios, *Bb*:*Ee*::*Ab* × *PB*:*Ae* × *PE*. Now let *Pb* return into its former place *PB*, and the

*Source: This translation by J. Stewart (1745) appears in D. J. Struik (ed.), *A Source Book in Mathematics, 1200-1800* (1969), 303-308. It is reprinted by permission of Harvard University Press, Copyright © 1969 by the President and Fellows of Harvard College. In this introduction Newton presents the characteristic triangle in number 5 and defines a tangent as the limit of a secant.

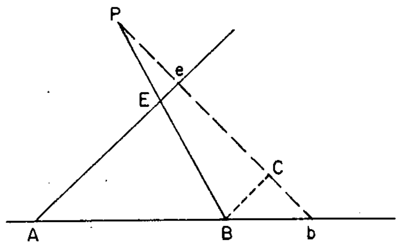

Fig. 3

evanescent augment Bb will be to the evanescent augment Ee as $AB \times PB$ to $AE \times PE$; and therefore the fluxion of the right line AB is to the fluxion of the right line AE in the same ratio.

10. Hence if the revolving right line PB cut any curve lines given in position in the points B and E, and the right lines AB, AE now becoming moveable, touch these curves in the points of section B and E: the fluxion of the curve, which the right line AB touches, shall be to the fluxion of the curve, which the right line AE touches, as $AB \times PB$ to $AE \times PE$. The same thing would happen if the right line PB perpetually touched any curve given in position in the moveable point P.

11. *Let the quantity x flow uniformly, and let it be proposed to find the fluxion of* x^n.

In the same time that the quantity x, by flowing, becomes $x + o$, the quantity x^n will become $(x + o)^n$, that is, by the method of infinite series, $x^n + nox^{n-1} + \frac{n^2-n}{2} oox^{n-2}$ + etc. And the augments o and $nox^{n-1} + \frac{n^2-n}{2} oox^{n-2}$ + etc. are to one another as 1 and $nx^{n-1} + \frac{n^2-n}{2} ox^{n-2}$ + etc.

Now let these augments vanish, and their ultimate ratio will be 1 to nx^{n-1}.

12. By like ways of reasoning, the fluxions of lines, whether right or curve in all cases, as likewise the fluxions of superficies, angles, and other quantities, may be collected by the method of *prime* and *ultimate* ratios. Now to institute an analysis after this manner in finite quantities and investigate the *prime* or *ultimate* ratios of these finite quantities when in their nascent or evanescent state, is consonant to the geometry of the ancients: and I was willing to show that, in the method of fluxions, there is no necessity of introducing figures infinitely small into geometry. Yet the analysis may be performed in any kind of figures, whether finite or infinitely small, which are imagined similar to the evanescent figures; as likewise in these figures, which, by the method of indivisibles, use to be reckoned as infinitely small, provided you proceed with due caution.

From the fluxions to find the fluents, is a much more difficult problem, and the first step of the solution is equivalent to the quadrature of curves; concerning which I wrote what follows some considerable time ago.

13. In what follows I consider indeterminate quantities as increasing or decreasing by a continued motion, that is, as flowing forwards, or backwards, and I design them by the letters z, y, x, v, and their fluxions or celerities of increasing I denote by the same letters pointed $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$. There are likewise fluxions or mutations more or less swift of these fluxions, which may be called the second fluxions of the same quantities z, y, x, v, and may be thus designed $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$: and the first fluxions of these last, or the third fluxions of z, y, x, v, are thus denoted $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$: and the fourth fluxions thus $\ddddot{z}$, $\ddddot{y}$, $\ddddot{x}$, $\ddddot{v}$. And after the same manner that $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$ are the fluxions of the quantities $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$, and these the fluxions of the quantities $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$; and these last fluxions of the quantities z, y, x, v: so the quantities z, y, x, v may be considered as the fluxions of others, which I shall design thus z', y', x', v'; and these as the fluxions of others z'', y'', x'', v''; and these last still as the fluxions of others z''', y''', x''', v'''. Therefore z'', z', z, $\dot{z}$, $\ddot{z}$, $\dddot{z}$, $\ddddot{z}$, etc. design a series of quantities whereof every one that follows is the fluxion of the one immediately preceding, and every one that goes before, is a flowing quantity having that which immediately succeeds, for its fluxion. The like is the series $\sqrt{az-zz}''$, $\sqrt{az-zz}'$, $\sqrt{az-zz}$, $\dot{\sqrt{az-zz}}$, $\ddot{\sqrt{az-zz}}$, $\dddot{\sqrt{az-zz}}$; as likewise the series $\frac{az+zz}{a-z}''$, $\frac{az+zz}{a-z}'$, $\frac{az+zz}{a-z}$, $\dot{\frac{az+zz}{a-z}}$, $\ddot{\frac{az+zz}{a-z}}$, $\dddot{\frac{az+zz}{a-z}}$, etc.

14. And it is to be remarked that any preceding quantity in these series is as the area of curvilinear figure of which the succeeding is the rectangular ordinate, and the abscis is z: as $\sqrt{ax-zz}'$ the area of a curve, whose ordinate is $\sqrt{az-zz}$, and absciss z. The design of all these things will appear in the following propositions.

PROPOSITION I
PROBLEM I

15. *An equation being given involving any number of flowing quantities, to find the fluxions.*[1]

Solution. Let every term of the equation be multiplied by the index of the power[2] of every flowing quantity that it involves, and in every multiplication change the side or root of the power into its fluxion, and the aggregate of all the products with their proper signs, will be the new equation.

16. *Explication.* Let a, b, c, d, etc. be determinate and invariable quantities, and let any equation be proposed involving the flowing quantities z, y, x, etc. as $x^3 - xy^2 + a^2z - b^3 = 0$. Let the terms be first multiplied by the indexes of the powers of x, and in every multiplication for the root, or x of one dimension write $\dot{x}$, and the sum of the factors will be $3\dot{x}x^2 - \dot{x}y^2$. Do the same in y, and there arises $-2xy\dot{y}$. Do the same in z, and there arises $aa\dot{z}$. Let the sum of these products be put equal to nothing, and you'll have the equation $3\dot{x}x^2 - \dot{x}y^2 - 2xy\dot{y} + aa\dot{z} = 0$. I say the relation of the fluxions is defined by this equation.

17. *Demonstration.* For let o be a very small quantity, and let $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ be the moments, that is the momentaneous synchronal increments of the quantities z, y, x. And if the flowing quantities are just now z, y, x, then after a moment of time, being increased by their increments $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, these quantities shall become $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$: which being wrote in the first equation for z, y, and x, give this equation $x^3 + 3x^2o\dot{x} + 3xoo\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xo\dot{y}y - 2\dot{x}o^2\dot{y}y - xo^2\dot{y}\dot{y} - \dot{x}o^3\dot{y}\dot{y} + a^2z + a^2o\dot{z} - b^3 = 0$.

Subtract the former equation from the latter, divide the remaining equation by o, and it will be $3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3o^2 - \dot{x}y^2 - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} - \dot{x}o^2\dot{y}\dot{y} + a^2\dot{z} = 0$. Let the quantity o be diminished infinitely, and neglecting the terms which vanish, there will remain $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Q.E.D.

18. *A fuller explication.* After the same manner if the equation were $x^3 - xy^2 + aa\sqrt{ax-y^2} - b^3 = 0$, thence would be produced $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + aa\dot{\sqrt{ax-y^2}} = 0$. Where if you would take away the fluxion $\dot{\sqrt{ax-y^2}}$, put $\sqrt{ax-y^2} = z$, and it will be $ax - y^2 = z^2$, and by this proposition $a\dot{x} - 2\dot{y}y = 2\dot{z}z$, or $\frac{a\dot{x}-2\dot{y}y}{2z} = \dot{z}$, that is $\frac{a\dot{x}-2\dot{y}y}{2\sqrt{ax-yy}} = \dot{\sqrt{ax-yy}}$. And thence $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + \frac{a^3\dot{x}-2a^2\dot{y}y}{2\sqrt{ax-yy}} = 0$.

19. And by repeating the operation, you proceed to second, third, and subsequent fluxions. Let $zy^3 - z^4 + a^4 = 0$ be an equation proposed, and by the first operation it becomes $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$; by the second $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 4\ddot{z}z^3 - 12\dot{z}^2z^2 = 0$, by the third, $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\dot{y}\ddot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$.

20. But when one proceeds thus to second, third, and following fluxions, it is proper to consider some quantity as flowing uniformly, and for its first fluxion to write unity, for the second and subsequent ones, nothing. Let there be given the equation $zy^3 - z^4 + a^4 = 0$,

as above; and let z flow uniformly, and let its fluxion be unity: then by the first operation it shall be $y^3 + 3z\dot{y}y^2 - 4z^3 = 0$; by the second $6\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 12z^2 = 0$; by the third $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z = 0$.

But in equations of this kind it must be conceived that the fluxions in all the terms are of the same order, i.e., either all of the first order $\dot{y}$, $\dot{z}$; or all of the second $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$; or all of the third $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, etc. And where the case is otherwise the order is to be completed by means of the fluxions of a quantity that flows uniformly, which fluxions are understood. Thus the last equation, by completing the third order, becomes $9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z\dot{z}^3 = 0$.[3]

NOTES

1. Newton prefers to differentiate equations, but later also differentiates functions, often given as areas.

2. [*Footnote by the translator, Stewart*] The word translated here *power* is *dignitas*, dignity, by which must be understood not only perfect, but also imperfect powers or surd roots, which are expressed in the manner of perfect powers, as is well known, by fractional indexes. In which sense $x^{1/2}$, $x^{2/3}$, etc. are powers; ½ and ⅔ their indexes, and x the side or root. I use the word power, because dignity is seldom used in English in this sense.

3. Newton insists on homogeneity, which requires that each term of the equation has the same number of "pricks."

Chapter VI
The Scientific Revolution at Its Zenith (1620–1720)

Section D
The Bernoullis

JAKOB BERNOULLI (1654–1705)

The Bernoulli family is the most famous in the history of mathematics. From the late 17th century to the present time it has contributed distinguished, and sometimes eminent, men of learning. The reputation of the Bernoulli's began with the careers of the brothers Jakob and Johann.

Jakob Bernoulli came from a thriving mercantile family in Basel, Switzerland. His father Nikolaus was a druggist and town magistrate; his mother Margaretha Schönauer was the daughter of a banker. They were a Protestant family whose ancestors had fled Antwerp in 1583 to escape the Catholic persecution of the Huguenots. Following the wishes of his father who wanted him to become a Protestant pastor, Jakob received a master of arts degree in philosophy from the University of Basel in 1671 and a licentiate in theology in 1676. However, he had other interests. As he stated in his motto *Invito patre sidera verso* ("against my father's will I study the stars"), he investigated astronomy and mathematics on his own.

The father's efforts to make Jakob Bernoulli a cleric were futile; a career in higher education was his goal upon graduation from the university. He began as a tutor in Geneva in late 1676 and then spent the next two years in France, familiarizing himself with the newly dominant Cartesian science, including the work of Nicolas Malebranche. Seeking more first-hand information on recent advances in the sciences, he took a second educational trip in 1681–1682. At this time he met the mathematician Jan Hudde in the Netherlands as well as the natural philosophers Robert Boyle and Robert Hooke in England. The main results of Bernoulli's early research were a theory of comets that later proved inadequate and a theory of gravity that his contemporaries regarded highly.

From 1683 on Jakob Bernoulli taught at the University of Basel and devoted his time to research in mathematics, astronomy, and mechanics. His careful study of the second edition of Franz Schooten's Latin translation of Descartes' *Géométrie* (1659–1661), John Wallis' *Arithmetica Infinitorum* ("The Arithmetic of Infinitesimals," 1656), and Isaac Barrow's *Lectiones Geometricae* ("Geometrical Lectures," 1664–1670) led him to the problem of infinitesimal geometry. In 1687 he was named professor of mathematics at the University of Basel. Already in 1683 his younger brother Johann had come to live with him and pursue university studies. Thereafter, the careers of the two men were closely linked, not always with happy results. To