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Continued after Index

The Arithmetic of Infinitesimals

John Wallis
1656

Translated from Latin to English with an Introduction
by
Jacqueline A. Stedall

Centre for the History of the Mathematical Sciences,
Open University

Springer

is, either a single mean proportional or the first of two, three, etc. In the same way, other distinct signs must be added which indicate, in the continued multiplications (of the given interpolated sequences), whether they increase by ones, or by twos, threes, etc. But all this, and whatever similar problems, must await more exact inquiry into these progressions, if mathematicians are of the opinion this should be admitted into arithmetic (and why less should be done, I do not see). It is sufficient for the purpose of the present work that we wish to indicate it in some way and to supply in plain words what is lacking in symbols. If, moreover, this method of notation thought out by me is less pleasing to mathematicians, I would as happily allow it to be changed to a way that they show more appropriate.

Howsoever this may be, I must indeed acknowledge that I am still unable to supply formulae of this kind for the odd sequences as for the even sequences in the table; nor for the odd places in the odd sequences (though I have now shown the ratios of those to each other) according to any method of notation (that I know yet accepted). And although in those above, often by fortune and by breaking paths never, as far as I know, trodden before, I have discovered some of the hoped for conclusions, I could scarcely, however, (for the reasons already shown) have dared to hope that likewise here also everything would come out as wished. If, by chance, anyone else from here on treading in my footsteps arrives at length at what it was not given to me to arrive at (for I would not wish to proclaim to the skilled the limits of all other methods in the same way as for mine), and discover more useful methods of expressing those same quantities, I would certainly not bear any ill will. In the meantime I believe it will be by no means unwelcome to mathematicians that I have offered some new light, not (as I judge it) wholly to be disparaged, on the obscurity of problems concerning the quadrature of the circle, and to have expressed that in numbers as far as the nature of numbers allows.

What we have already found, moreover, it may also be pleasing to set out in some following Propositions, in a form a little changed. And first indeed it may be signified as closely as one wishes by whole numbers, and afterwards also by straight lines.

$\square = \frac{4}{\pi}$

PROPOSITION 191

Problem

It is proposed to inquire, what is the value of the term □ (in the table of Proposition 189), as closely as one wishes using whole numbers.

That the thing may come out more easily, the terms of the progression (the same produced again) $\frac{1}{2}\square$, 1, □, $\frac{3}{2}$, $\frac{4}{3}\square$, $\frac{3\times5}{2\times4}$, $\frac{4\times6}{3\times5}\square$, $\frac{3\times5\times7}{2\times4\times6}$, etc. may be called $\alpha, a, \beta, b, \gamma, c, \delta, d$, etc.

Moreover, $1:2=\alpha:\beta$, and $2:3=a:b$, and $3:4=\beta:\gamma$, and $4:5=b:c$, and $5:6=\gamma:\delta$, and $6:7=c:d$.

That is, $\frac{\beta}{\alpha}=\frac{2}{1}$, $\frac{b}{a}=\frac{3}{2}$, $\frac{\gamma}{\beta}=\frac{4}{3}$, $\frac{c}{b}=\frac{5}{4}$, $\frac{\delta}{\gamma}=\frac{6}{5}$, $\frac{d}{c}=\frac{7}{6}$, etc.
Therefore (since the multiplying ratios continually decrease) we will have

$$\frac{\beta}{a}\text{ is}\begin{cases}\text{the lesser of both}^{77} & \frac{a}{\alpha}\times\frac{\beta}{a}=\frac{\beta}{\alpha}=\frac{2}{1}, \quad \text{therefore less than } \sqrt{\tfrac{2}{1}}=\sqrt{1\tfrac{1}{1}}\\ \text{the greater of both} & \frac{\beta}{a}\times\frac{b}{\beta}=\frac{b}{a}=\frac{3}{2}, \quad \text{therefore greater than } \sqrt{\tfrac{3}{2}}=\sqrt{1\tfrac{1}{2}}\end{cases}$$

$$\text{and therefore } \beta=a\times\frac{\beta}{a}=\square\text{ is}\begin{cases}\text{less than} & 1\sqrt{2}=1\sqrt{1\tfrac{1}{1}}\\ \text{greater than} & 1\sqrt{\tfrac{3}{2}}=1\sqrt{1\tfrac{1}{2}}\end{cases}$$

In the same way

$$\frac{\gamma}{b}\text{ is}\begin{cases}\text{the lesser of both} & \frac{b}{\beta}\times\frac{\gamma}{b}=\frac{\gamma}{\beta}=\frac{4}{3}, \quad \text{therefore less than } \sqrt{\tfrac{4}{3}}=\sqrt{1\tfrac{1}{3}}\\ \text{the greater of both} & \frac{\gamma}{b}\times\frac{c}{\gamma}=\frac{c}{b}=\frac{5}{4}, \quad \text{therefore greater than } \sqrt{\tfrac{5}{4}}=\sqrt{1\tfrac{1}{4}}\end{cases}$$

$$\text{and therefore } \gamma=b\times\frac{\gamma}{b}=\frac{4}{3}\square\text{ is}\begin{cases}\text{less than} & \frac{3}{2}\times\sqrt{1\tfrac{1}{3}}\\ \text{greater than} & \frac{3}{2}\times\sqrt{1\tfrac{1}{4}}\end{cases}$$

that is, □ is less than $\frac{3\times3}{2\times4}\times\sqrt{1\frac{1}{3}}$, greater than $\frac{3\times3}{2\times4}\times\sqrt{1\frac{1}{4}}$

And (by the same reasoning)

$$\delta=c\times\frac{\delta}{c}=\frac{4\times6}{3\times5}\square\text{ is}\begin{cases}\text{less than} & \frac{3\times5}{2\times4}\times\sqrt{1\tfrac{1}{5}}\\ \text{greater than} & \frac{3\times5}{2\times4}\times\sqrt{1\tfrac{1}{6}}\end{cases}$$

that is, □ is less than $\frac{3\times3\times5\times5}{2\times4\times4\times6}\times\sqrt{1\frac{1}{5}}$, greater than $\frac{3\times3\times5\times5}{2\times4\times4\times6}\times\sqrt{1\frac{1}{6}}$

And (continuing the work in this way according to the rules of the table) it will be found that

$$\square\text{ is}\begin{cases}\text{less than} & \frac{3\times3\times5\times5\times7\times7\times9\times9\times11\times11\times13\times13}{2\times4\times4\times6\times6\times8\times8\times10\times10\times12\times12\times14}\times\sqrt{1\tfrac{1}{13}}\\ \text{greater than} & \frac{3\times3\times5\times5\times7\times7\times9\times9\times11\times11\times13\times13}{2\times4\times4\times6\times6\times8\times8\times10\times10\times12\times12\times14}\times\sqrt{1\tfrac{1}{14}}\end{cases}$$

77 Wallis's argument here is that β/a is the smaller of the two quantities a/α and β/a (because of the decreasing ratio), and is therefore less than the square root of their product. Wallis does not make himself entirely clear, and Christiaan Huygens was puzzled by this part of the argument, and failed to understand why Wallis went on to take a square root; see Huygens to Wallis, [11]/21 July 1656, Beeley and Scriba 2003, 189–192.

And so on as far as one likes. In such a way, that is, that the numerator of the fraction arises from continually multiplying odd numbers 3, 5, 7, etc. placed twice, but the denominator from continually multiplying even numbers 2, 4, 6, etc. also placed twice, except the first and last, which are put only once; and finally all that ratio, or fraction, thus formed, is multiplied by the square root of 1 increased by some fraction of itself, namely that which has as its denominator the last of the odd numbers in the continued multiplication, if we seek a number too large, or of the evens, if we seek a number too small.

And by this method it may be done as far as one likes until the difference between the greater and the smaller becomes less than any assigned quantity (which, therefore, if one supposes the operation continued infinitely, will at last disappear). Which indeed, in case it is needed, will be demonstrated here.

Thus, as has already been said of the numbers in the continued multiplication, the greatest of the evens (that is, the final factor of the denominator) may be called z, and therefore the greatest of the odds (that is, the final factor of the numerator) will be $z-1$ (that is, the other less one). Therefore (since the same multiplier is combined with both) the number too large to the number too small will be as $\sqrt{1\frac{1}{z-1}}$ to $\sqrt{1\frac{1}{z}}$, that is, as the final surd number in the former to the final surd number in the latter), that is, as $\sqrt{\frac{z}{z-1}}$ to $\sqrt{\frac{z+1}{z}}$, that is, as $\sqrt{\frac{z^2}{z-1}}$ to $\sqrt{(z+1)}$ that is as $\sqrt{z^2}=z$ to $\sqrt{(z^2-1)}$. Moreover it may happen (by increasing the quantity z as needed) that the difference between the roots $\sqrt{z^2}$ and $\sqrt{(z^2-1)}$, that is, $z-\sqrt{(z^2-1)}$, becomes less than any assigned quantity (as is known, and was also said elsewhere by me at Proposition 39 of *On conic sections*). And therefore the number too large exceeds the number too small by a fraction less than any assigned quantity.[78] Which was to be proved.

Since, moreover, as is clear from what has been said, by increasing the number z infinitely, the number too large exceeds the number too small by a fraction less than any assigned quantity, the differences between them (and therefore of either from the true quantity) will be infinitely small, that is, nothing.

Further, since the number z is thus increased infinitely, that fractional part of 1 adjoined to it will be infinitely small; it will be $\sqrt{1\frac{1}{z}}$ or $\sqrt{1\frac{1}{z-1}}$, which amounts to the same thing therefore as $\sqrt{1}$ or 1 (on account of the vanishing infinitely small part), which by multiplication changes nothing. We say that the fraction $\frac{3\times 3\times 5\times 5\times 7\times 7 etc.}{2\times 4\times 4\times 6\times 6\times 8 etc.}$ or $\frac{9\times 25\times 49\times 81 etc.}{8\times 24\times 48\times 80 etc.}$ continued infinitely is itself precisely the required number □, and the ratio of 1 to this is that of the circle to the square of its diameter. Or (if this is more pleasing), as the denominator of that fraction is to the numerator, so we may say is the circle to the square of its diameter. And, as the numerator is to the denominator, so is the square to the circle. That is, as the product of the continued multiplication $9\times 25\times 49\times 81$ etc. (squares of odd numbers) to the product of $8\times 24\times 48\times 80$ etc. (the same squares decreased by one), continued infinitely.

[78] Wallis's proof has interesting elements of a later limit argument, but is incomplete. His argument that $z-\sqrt{(z^2-1)}$ can be made less than any assigned quantity is correct; he has ignored, however, the fact that this quantity is multiplied by a fraction that increases with each new pair of multipliers. The convergence of the fraction therefore depends on the properties of not one but two infinite processes.

Moreover, if some more curious person inquires how far that continued multiplication must be continued until at last that given difference, or less than that, is arrived at, or so that the number too large exceeds the number too small, by however small a part of itself (or not even that), that will be investigated by this method.

Let the greater quantity be called m, the smaller n, and let their difference, that part however small, thus $\frac{a}{b}m = m-n$, and let it be inquired how far the work must be continued, that is, what will be the number z, the greatest (simple) multiplier that produces that difference (or even less than it).

Since therefore $m-n=\frac{a}{b}m$, we will have $n=m-\frac{a}{b}m$, and $m:n=m:m-\frac{a}{b}m=\frac{b}{b}m:\frac{b-a}{b}m=b:b-a=z:\sqrt{(z^2-1)}$ (by the method demonstrated). Therefore $b\sqrt{(z^2-1)}=bz-az$. And (squaring everywhere) $b^2z^2-b^2=b^2z^2+a^2z^2-2abz^2$. And then (deleting b^2z^2 everywhere and transposing the rest) $2abz^2-a^2z^2=b^2$. And finally (dividing everywhere) $z^2=\frac{b^2}{2ab-a^2}$. Therefore the square root of this number (if it is an even number), or at least (if it is either a fraction or a surd or an odd number) the even number next greater than that root, will be the greatest of the multipliers that arrives at the assigned difference or certainly less than that. Which was to be investigated.

The same another way

After this our description of that quantity □, we may also add another, which I have received from that most noble person and very skilled geometer, Lord William Viscount and Baronet Brouncker.

Since I showed him some of my progressions, and indicated by what rule they proceeded, meanwhile asking him to indicate in what form he thought that quantity might usefully be described. That Noble Gentleman, having thought it over himself, judged by a method of infinites of his own that the same quantity could be most conveniently described in this form:

$$\square = 1\cfrac{1}{2\cfrac{9}{2\cfrac{25}{2\cfrac{49}{2\cfrac{81}{2}\ \text{etc.}}}}}$$

That is, if to one there is added a fraction that has a denominator continually broken, by the rule that the numerators of each small fraction are 1, 9, 25, etc., squares of odd numbers 1, 3, 5, etc., but the denominators everywhere 2 with an adjoined fraction, and thus infinitely. Adding this at the same time, that, wherever at length it pleases one to stop, instead of the final 2 with the fraction afterwards cut off, there may be put (according to the place where