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ORDERINGS FOR INCOMPLETE FACTORIZATION PRECONDITIONING OF NONSYMMETRIC PROBLEMS*

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Abstract. Numerical experiments are presented whereby the effect of reorderings on the convergence of preconditioned Krylov subspace methods for the solution of nonsymmetric linear systems is shown. The preconditioners used in this study are different variants of incomplete factorizations. It is shown that certain reorderings for direct methods, such as reverse Cuthill–McKee, can be very beneficial. The benefit can be seen in the reduction of the number of iterations and also in measuring the deviation of the preconditioned operator from the identity.

Key words. linear systems, nonsymmetric matrices, reorderings, permutation of sparse matrices, preconditioned iterative methods, incomplete factorizations, Krylov subspace methods

AMS subject classifications. Primary, 65F10, 65N22, 65F50; Secondary, 15A06

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1. Introduction.

In this paper, we study experimentally how different reorderings affect the convergence of Krylov subspace methods for nonsymmetric systems of linear equations when incomplete LU factorizations are used as preconditioners. In other words, given a sparse linear system of equations $Av = b$, where v and b are n-dimensional vectors, we consider symmetric permutations of the matrix A, i.e., of the form P^TAP, and then solve the equivalent system $P^TAPw = P^Tb$, with $v = Pw$, by way of some preconditioned iterative method. Our focus is on linear systems arising from the discretization of second order partial differential equations, which often are structurally symmetric (or very nearly so) and have a zero-free diagonal. For these matrices, it is usually possible to carry out an incomplete factorization without pivoting for stability (that is, choosing the pivots from the main diagonal). Such properties are preserved under symmetric permutations of A, but not necessarily under nonsymmetric ones. Hence, we restrict our attention to symmetric permutations only. We stress the fact that very different conclusions may hold for matrices which are structurally far from being symmetric, although we have little experience with such problems. If A is structurally symmetric, the reorderings are based on the (undirected) graph associated with the structure of A; otherwise, the structure of $A + A^T$ is used. We consider several iterative methods for nonsymmetric systems, including GMRES [43], Bi-CGSTAB [48], and transpose-free QMR (TFQMR) [28]; for a description of these, as well as a description of incomplete factorizations, see, e.g., [3], [42].

In this paper, we mainly concentrate on orderings originally devised for matrix factorizations, i.e., those used to reduce fill-in in the factors; see, e.g., [18] or [29]. We want to call attention to the fact that a permutation of the variables (and equations)

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using reordering methods designed for direct solvers can have an important positive effect on the robustness and performance of preconditioned Krylov subspace methods when applied to nonsymmetric linear systems. This is especially the case if the matrices are far from symmetric. This observation, although not completely new, is not fully appreciated, to the point that some authors have concluded that direct solver reorderings should not be used with preconditioned iterative methods; see section 2. It is hoped that the present study will contribute to a reassessment of direct solver reorderings in the context of incomplete factorization preconditioning. Furthermore, we hope that the evidence of our experiments can set the stage for more widespread use of these reorderings.

Many papers have been written on the effect of permutations on the convergence of preconditioned Krylov subspace methods. The main contributions are surveyed, together with some of our observations, in section 2. In section 3 we present our numerical experiments and comment on those results. Finally, in section 4 we present our conclusions.

2. Overview of the literature. The influence of reorderings on the convergence of preconditioned iterative methods has been considered by a number of authors. Several of these papers are concerned with symmetric problems only [8], [12], [14], [21], [24], [27], [36], [37], [38], [45], [49]. In this context, Duff and Meurant [21] have performed a very detailed study of the effects of reorderings for preconditioned conjugate gradients, i.e., in the symmetric positive definite case. Based on their extensive experiments, they concluded that the number of iterations required for convergence with direct solver reorderings is usually about the same as, and sometimes considerably higher than, with the natural (lexicographic) ordering. An important observation in [21] is that the number of conjugate gradient iterations is not related to the number of fill-ins discarded in the incomplete factorization (as conjectured by Simon [44]) but is almost directly related to the norm of the residual matrix $R = A - \bar{L}\bar{L}^T$, where $\bar{L}$ is an incomplete Cholesky factor of A; see [1] for a rigorous derivation of this result under appropriate conditions. Throughout the paper we abuse the notation and use A to denote both the original matrix and the permuted one. Similarly, $\bar{L}$ and $\bar{U}$ refer to the incomplete factors of A or those of P^TAP, depending on the context.

As it can be seen in the experiments in section 3, for the test matrices that are nearly symmetric, our observations are in agreement with those of Duff and Meurant: the reorderings have no positive effect on the convergence of the preconditioned Krylov methods. On the other hand, for the highly nonsymmetric test matrices, i.e., when the nonsymmetric part is large, we conclude that reorderings can indeed make a big difference. Permutations that appear to be ineffective for the (nearly) symmetric case turn out to be very beneficial, often improving the robustness and performance of the preconditioned iteration dramatically. (It is worth emphasizing that in [21], only symmetric problems are considered.)

In addition, we will see that for problems that are strongly nonsymmetric and/or are far from being diagonally dominant, the norm of the residual matrix R alone is usually not a reliable indicator of the quality of the corresponding preconditioner. It has been pointed out, e.g., in [11], that a more revealing measure of the quality of the preconditioner can be obtained by considering the Frobenius norm of the deviation of the preconditioned matrix from the identity, i.e., $\|I - A(\bar{L}\bar{U})^{-1}\|_F$. Note that this quantity is equal to $\|R(\bar{L}\bar{U})^{-1}\|_F$. Even if R is small in norm, it could happen that $(\bar{L}\bar{U})^{-1}$ has very large entries, resulting in a large deviation of the preconditioned matrix from the identity. As a result, the preconditioned iteration fails to converge. A notable example of this phenomenon occurs when convection-dominated convection-diffusion equations are discretized with centered finite differences. When the natural (lexicographic) ordering is used, the incomplete triangular factors resulting from a no-fill ILU factorization tend to be very ill conditioned, even if the coefficient matrix itself is well conditioned. Allowing more fill-in in the factors, e.g., using ILU(1) or ILUT instead of ILU(0), will solve the problem in some cases but not always. This kind of instability of ILU factorizations was first noticed by van der Vorst [47] and analyzed in detail by Elman [25].

We will see that in some cases, reordering the coefficient matrix before performing the incomplete factorization can have the effect of producing stable triangular factors, and hence more effective preconditioners. Generally speaking, $\|R\|_F$ and $\|R(\bar{L}\bar{U})^{-1}\|_F$ do not contain enough information to describe in a quantitative fashion the behavior of incomplete factorizations. In particular, it is not possible in general to establish comparisons between incomplete factorizations based on these quantities alone. This is not surprising, considering that for general nonsymmetric problems it is not known how to predict the rate of convergence of iterative solvers. In practice, however, one can expect that very ill-conditioned incomplete $\bar{L}$ and $\bar{U}$ factors will result in a poor preconditioner. As suggested in [11], an inexpensive way of detecting this ill-conditioning is by computing $\|(\bar{L}\bar{U})^{-1}e\|_\infty$, where e denotes a vector of all ones. This is only a lower bound for $\|(\bar{L}\bar{U})^{-1}\|_\infty$ but is quite useful in practice.

The effects of permutations on preconditioned Krylov subspace methods for nonsymmetric problems have been considered in [9], [13], [15], [16], [17], [22], [33], [44], [46]. Some authors have concluded that the reorderings designed for sparse direct solvers are not recommended for use with preconditioned iterative methods; see, e.g., [14], [33], [44]. Simon [44] used quotient minimum degree and nested dissection [29] in conjunction with an ILU preconditioner for some oil simulation problems and found essentially no improvement over the original ordering. He wondered if this strategy would be advantageous for other kinds of problems. Similar conclusions were reached by Langtangen [33], who applied a minimum degree reordering with ILU(0) preconditioning of matrices arising from a Petrov–Galerkin formulation for convection-diffusion equations. It should be mentioned that neither the problems considered by Simon nor those considered by Langtangen exhibited any kind of instability of the incomplete triangular factors and that most of those problems can be regarded as fairly easy to solve, at least by today's standards. Dutto [22], in the context of a specific application (solving the compressible Navier–Stokes equations with finite elements on unstructured grids), was possibly the first to observe that minimum degree and other direct solver reorderings can have a positive effect on the convergence of GMRES with ILU(0) preconditioning. This is mostly consistent with some of our own experiments reported here.

In the context of oil reservoir simulations, reverse Cuthill–McKee and some variants of it were found to perform satisfactorily for symmetric, strongly anisotropic problems due to the fact that these orderings are relatively insensitive to anisotropies [4], [12], [49]. Numerical experiments indicating that the D2 diagonal ordering (which is a special case of Cuthill–McKee) for ILU(k) preconditioning of certain nonsymmetric problems defined on rectangular grids can be superior to the natural ordering were reported in [5], but this observation did not receive the attention it deserved. This may be due in part to the fact that the authors use a terminology that is peculiar to the field of reservoir simulation. A short paragraph mentioning that level set reorderings (like reverse Cuthill–McKee) can be useful for preconditioned iterative methods can be found in [42], and similar remarks are no doubt found elsewhere in the literature. In spite of this, reorderings for direct methods are still widely regarded as ineffective (or even bad) for preconditioned iterative methods.

We also mention the minimum discarded fill (MDF) algorithm (see [13], [14]), which takes into account the numerical values of the entries of A. This method can be very effective, but it is often too expensive to be practical, except for rather simple problems.

In addition to reorderings which were originally designed for direct methods, one can use the permutation produced by the algorithm TPABLO [10], which also uses the magnitude of the entries of the matrix. This algorithm produces a permuted matrix with dense diagonal blocks, while the entries outside the blocks on the diagonal have magnitude below a prescribed threshold. Hence, like MDF, the TPABLO reordering is based both on graph information and on the numerical values. The original motivation for the TPABLO algorithm was to produce good block diagonal preconditioners, and also blocks for the treatment of certain Markov chain problems [10]; see also [23]. It turns out that this reordering is also useful for point incomplete factorizations, where it is often better than the natural ordering [6] and some of the reorderings considered in this paper [7]. However, for most cases treated in this paper, the performance of TPABLO is inferior to that of some of the reorderings designed for direct methods, and thus we do not report results with it here. TPABLO might prove useful in the context of *block* incomplete factorizations, but this topic is outside the scope of the present paper.

Finally, a number of papers have considered other kinds of reorderings, such as those motivated by parallel computing (e.g., multicoloring; see [8], [15], [27], [37]) and reorderings based on the physics underlying the discrete problem being solved [9]. Such reorderings can be very useful in practice, but they are strongly architecture and problem specific.

3. Numerical experiments.

In this section we show, by means of numerical experiments, that direct solver reorderings can be very beneficial when solving *difficult* nonsymmetric linear systems (obviously, at best, little gains can be expected from reordering problems which are easily solved with the original ordering). The results reported here are a representative selection from a large number of experiments with nonsymmetric matrices arising from the numerical solution of partial differential equations. In the first subsection we focus on an important class of problems (convection-diffusion equations discretized with finite differences), while in the second we present a selection of results for matrices from a variety of applications. In the last subsection we investigate reasons why reordering improves the performance of the preconditioners. All the experiments were performed on a Sun Ultra SPARC workstation using double precision arithmetic. Codes were written in standard Fortran-77.

3.1. Convection-diffusion equations.

A source of linear systems which can be challenging for iterative methods is the following partial differential equation in the open unit square $\Omega = (0,1) \times (0,1)$:

$$-\varepsilon \Delta u + \frac{\partial e^{xy} u}{\partial x} + \frac{\partial e^{-xy} u}{\partial y} = g \tag{1}$$

with homogeneous Dirichlet boundary conditions. Equation (1) has been repeatedly used as a model problem in the literature; see, e.g., [39]. The problem is discretized using centered differences for both the second order and first order derivatives with grid size $h = 1/33$, leading to a block tridiagonal linear system of order 1024 with 4992 nonzero coefficients. While this is a small problem size, it exhibits the features we wish to address here. Numerical experiments were also performed with finer grids and with a three-dimensional analogue of the same PDE, and the results obtained were similar to those reported in Tables 1–4 below. For the right-hand side we used a random vector. Results similar to those in Tables 1–4 were obtained with other choices of b. In all our experiments we used $v_0 = 0$ as initial guess and we stopped the iterations when the 2-norm of the (unpreconditioned) residual $b - Av_k$ had been reduced to less than 10^{-4} or when a maximum number of iterations was reached. The parameter $\varepsilon > 0$ controls the difficulty of the problem—the smaller ε is, the harder it is to solve the discrete problem by iterative methods. For our experiments, we generated 10 linear systems of increasing difficulty, corresponding to $\varepsilon^{-1} = 100, 200, \ldots, 1000$. The coefficient matrix A becomes more nonsymmetric (and less diagonally dominant) as ε decreases. If we denote by S and T the symmetric and the skew-symmetric part of A, respectively, then for $\varepsilon^{-1} = 100$ we have $\|S\|_F = 1.44$ and $\|T\|_F = 1.06$; as ε gets smaller, the norm of T remains unchanged, whereas the norm of S decreases. For $\varepsilon^{-1} = 1000$, we have $\|S\|_F = 0.154$. We point out incidentally that these quantities are invariant under symmetric permutations, so the departure from symmetry cannot be altered simply by reordering the matrix; however, we will see that from the point of view of incomplete factorization preconditioning, some orderings are less sensitive to the departure from symmetry than others.

There are many possible ways of implementing the (reverse) Cuthill–McKee and minimum degree reorderings. We used Liu's multiple minimum degree algorithm [34], and for the Cuthill–McKee reorderings, we used an implementation which chooses a pseudoperipheral node as starting node and sorts nodes in the same level set by increasing degree [29]. Other strategies are possible as well, and different choices may lead to somewhat different results. We note here that we experimented also with one-way and nested dissection reorderings, and for discrete convection-diffusion problems the results found were comparable to those obtained with the multiple minimum degree reordering, both from the point of view of fill-in in the incomplete factors and from the point of view of convergence rates. For this reason we do not show these results in the tables.

We report the results of experiments with the following accelerators: Bi-CGSTAB, TFQMR, and GMRES with restart parameter $m = 20$. The preconditioners used were standard incomplete factorizations based on levels of fill (ILU(0) and ILU(1); see [35]) and Saad's dual threshold ILUT; see [41], [42]. For ILUT, we used two different sets of parameters, $(10^{-2}, 5)$ and $(10^{-3}, 10)$. The latter results in a very powerful but expensive preconditioner, containing up to five times the number of nonzeros in A. Right preconditioning was used in all cases.

In Tables 1–4 we present the number of iterations for the different orderings and the three Krylov subspace methods used in this study. In these tables, and in the ones that follow, n/o stands for natural (or original) ordering, CM for Cuthill–McKee, RC for reverse Cuthill–McKee, and MD for multiple minimum degree. The symbol † indicates that convergence was not achieved in 250 iterations for Bi-CGSTAB and TFQMR, which require two matrix-vector products and applications of the preconditioner per iteration, and in 500 iterations for GMRES(20), which requires only one matrix-vector product and preconditioner application per iteration. In each case the number in bold indicates the run which required the least amount of work. This is not always the same as the run which required the least number of iterations because,

TABLE 1
Number of iterations for different orderings, preconditioner ILU(0).

	Bi-CGSTAB				GMRES(20)				TFQMR			
ε^{-1}	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
100	**6**	7	7	23	**9**	**9**	**9**	61	8	**7**	**7**	24
200	**12**	14	13	23	**14**	**14**	**14**	74	**11**	**11**	**11**	24
300	**17**	19	18	28	27	**26**	27	79	**15**	**15**	**15**	30
400	28	32	**26**	34	48	48	**47**	80	**27**	**27**	**27**	36
500	63	56	59	**42**	92	**79**	89	85	50	57	52	**43**
600	96	106	79	**49**	140	140	139	**90**	86	108	109	**51**
700	160	170	145	**51**	320	277	336	**94**	207	203	216	**61**
800	†	219	†	**59**	†	†	†	**101**	†	†	†	**68**
900	†	†	†	**57**	†	†	†	**110**	†	†	†	**71**
1000	†	†	†	**61**	†	†	†	**123**	†	†	†	**75**

TABLE 2
Number of iterations for different orderings, preconditioner ILU(1).

	Bi-CGSTAB				GMRES(20)				TFQMR			
ε^{-1}	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
100	**3**	**3**	**3**	10	**5**	**5**	**5**	14	**3**	**3**	**3**	11
200	6	**5**	**5**	11	9	**7**	**7**	16	7	**5**	6	11
300	9	**5**	**5**	15	13	**8**	**8**	22	10	7	**5**	15
400	12	**6**	**6**	21	17	**9**	10	32	13	8	**6**	22
500	18	**7**	**7**	28	22	**11**	**11**	45	15	9	**7**	27
600	28	**8**	**8**	39	34	**13**	**13**	61	19	9	**8**	34
700	38	**10**	**10**	44	41	**14**	**14**	73	22	**10**	**10**	39
800	68	**11**	12	47	61	**16**	**16**	74	32	**11**	12	44
900	142	**12**	16	55	96	**18**	**18**	89	38	**12**	**12**	44
1000	†	**16**	18	59	137	23	**22**	97	49	**14**	16	50

in general, different reorderings result in preconditioners with a different number of nonzeros (except, of course, for ILU(0), where the number of nonzeros is always equal to the number of nonzero entries in A). For this particular class of matrices, the amount of fill-in in the incomplete factors is often highest for the natural ordering. Cuthill–McKee and reverse Cuthill–McKee result in comparable fill-in (slightly less, on average, than with the natural ordering), while minimum degree produces the least amount of fill-in. For example, in the case $\varepsilon^{-1} = 100$, the ILUT$(10^{-2}, 5)$ factors contain 12907 nonzeros with the natural ordering, 10271 nonzeros with Cuthill–McKee, 10143 nonzeros with reverse Cuthill–McKee, and 9038 nonzeros for multiple minimum degree. As ε gets smaller, these values slowly increase (more or less uniformly for all orderings). For $\varepsilon^{-1} = 1000$ we have 14695 nonzeros for the natural ordering, 15023 for Cuthill–McKee, 14259 for reverse Cuthill–McKee, and 9348 for multiple minimum degree.

We now comment on the numerical results. We notice that for the moderately nonsymmetric problems (smaller values of ε^{-1}), the alternative permutations offer little or no advantage over the natural ordering. In particular, the Cuthill–McKee and reverse Cuthill–McKee reorderings produce nearly the same results as the natural ordering. It is known (see, e.g., [50, sections 5.5 and 5.6]) that for five-point stencils and no-fill factorizations, the Cuthill–McKee reorderings are equivalent to the natural ordering in the sense that the incomplete factors of the permuted matrix are just the permuted incomplete factors of the original matrix. Hence the ILU(0) preconditioners with the natural ordering and Cuthill–McKee reorderings are mathematically equivalent. This is true, however, only if the starting node is the same for both types of

Table 3
Number of iterations for different orderings, preconditioner ILUT$(10^{-2}, 5)$.

	Bi-CGSTAB				GMRES(20)				TFQMR			
ε^{-1}	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
100	2	1	**1**	4	3	2	**2**	6	2	**1**	2	5
200	3	1	**1**	5	4	2	**2**	10	3	1	**1**	8
300	3	2	**2**	11	6	3	**3**	17	3	**2**	3	12
400	4	2	**2**	21	7	3	**3**	32	4	2	**2**	23
500	5	2	**2**	30	9	3	**3**	50	6	3	**2**	33
600	6	2	**2**	113	11	4	**4**	†	6	3	**2**	201
700	8	3	**2**	†	13	4	**4**	†	9	3	**3**	†
800	11	3	**2**	†	17	5	**4**	†	11	3	**3**	†
900	†	3	**2**	†	†	6	**4**	†	†	4	**3**	†
1000	†	4	**3**	†	†	7	**5**	†	†	4	**3**	†

Table 4
Number of iterations for different orderings, preconditioner ILUT$(10^{-3}, 10)$.

	Bi-CGSTAB				GMRES(20)				TFQMR			
ε^{-1}	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
100	1	1	**1**	3	2	2	**2**	4	2	**1**	2	4
200	2	1	**1**	3	3	2	**2**	50	2	1	**1**	3
300	2	1	**1**	3	4	2	**2**	6	2	1	**1**	4
400	3	1	**1**	4	4	2	**2**	7	3	1	**1**	4
500	3	1	**1**	5	5	2	**2**	8	3	1	**1**	4
600	4	1	**1**	5	6	2	**2**	9	5	**1**	2	6
700	6	2	**2**	6	9	3	**3**	10	6	2	**2**	6
800	6	2	**2**	6	9	3	**3**	11	6	3	**2**	7
900	8	2	**2**	7	11	3	**3**	12	8	2	**2**	9
1000	9	2	**2**	7	13	3	**3**	13	10	**2**	3	8

orderings. For our implementation of (reverse) Cuthill–McKee, this is not the case in general. Hence, there are some differences in the number of iterations obtained. We note that these discrepancies are more pronounced for larger values of ε^{-1}, suggesting that the sensitivity of ILU(0) preconditioning to the choice of the starting node becomes stronger as the matrix becomes increasingly nonsymmetric and farther from being diagonally dominant.

For larger values of ε, minimum degree causes a serious degradation of the convergence rate, especially with ILU(0). Notice that the differences in behavior are not as pronounced with the ILUT preconditioners, which use a drop tolerance.

As the coefficient matrix becomes increasingly nonsymmetric, however, things change. While the number of iterations increases for all reorderings, the rate of increase is not the same for all reorderings, suggesting that some orderings are less sensitive than others to the degree of nonsymmetry. For ILU(0), the natural ordering and the Cuthill–McKee reorderings exhibit the worst degradation as ε decreases. The best performance is achieved with minimum degree, which is the only reordering which caused all three iterative solvers to converge on all problems. The situation is quite different with ILU(1) and the ILUT preconditioners. With ILU(1), the natural ordering performs poorly, minimum degree is only slightly better, but the Cuthill–McKee reorderings are both quite good. With ILUT$(10^{-2}, 5)$ minimum degree is very bad and the Cuthill–McKee orderings are both excellent. With ILUT$(10^{-3}, 10)$, which gives very good (but expensive) approximations of A, all orderings produce effective preconditioners, but the performance is particularly good with the Cuthill–McKee reorderings. Notice that reverse Cuthill–McKee is only slightly better than Cuthill–

McKee. It is worth mentioning that for the natural, Cuthill–McKee, and reverse Cuthill–McKee orderings the number of nonzeros in the ILUT$(10^{-3}, 10)$ factors is approximately 35% of the number of nonzeros in the *complete* LU factors computed with the same ordering; for multiple minimum degree, this proportion goes up to about 58%. For the more powerful preconditioners (ILU(1) and ILUT) the Cuthill–McKee reorderings give the best results for *all* values of ε. As far as the relative performance of the three Krylov subspace solvers is concerned, we observe that they are more or less equivalent for this particular class of problems, with GMRES(20) requiring fewer matrix-vector products and preconditioner applications than the other solvers in many cases.

This set of problems was generated using second order, centered difference approximations for both the second and first partial derivatives in (1). It is well known that for large values of h/ε, this discretization can become unstable. Alternative discretizations, such as those which use upwinding for the first order terms, do not suffer from this problem and give rise to matrices with very nice properties from the point of view of iterative solutions, such as diagonal dominance. However, this may not be true for nonuniform grids (see [30]), and, moreover, such approximations are only first order accurate and in many cases are unable to resolve fine features of the solution, such as boundary layers. In this case, as suggested in [31], a uniform coarse grid could be used to determine the region where the boundary layer is located (this corresponds to "wiggles" in an otherwise smooth solution). This would require solving linear systems such as those considered in the previous set of experiments. Subsequently, a local mesh refinement can be performed in the region containing the boundary layer. This solves the instability problem, and the approximation is still second order accurate (except at a few points on the interface between the coarse and the fine grids), but the resulting linear system, like the one corresponding to the uniform grid, can be quite challenging for iterative methods if convection is strong. Again, a simple reordering of the coefficient matrix can improve the situation dramatically. To illustrate this, we take the following example from Elman [26]. Consider the following partial differential equation in $\Omega = (0, 1) \times (0, 1)$:

$$-\Delta u - 2P\frac{\partial u}{\partial x} + 2P\frac{\partial u}{\partial y} = g, \tag{2}$$

where $P > 0$ and the right-hand side g and the boundary conditions are determined by the solution

$$u(x, y) = \frac{e^{2P(1-x)} - 1}{e^{2P} - 1} + \frac{e^{2Py} - 1}{e^{2P} - 1}.$$

This function is nearly identically zero in Ω except for boundary layers of width $\mathcal{O}(\delta)$ near $x = 0$ and $y = 1$, where $\delta = 1/2P$. A uniform coarse grid was used in the region where the solution is smooth, and a uniform fine grid was superimposed on the regions containing the boundary layers, so as to produce a stable and accurate approximation; see [26] for details.

We performed experiments with $P = 500$ and $P = 1000$. These values are considerably larger than those used in [26]. The resulting matrices are of order 5041 and 7921, with 24921 and 39249 nonzeros, respectively. The convergence criterion used was a reduction of the residual norm to less than 10^{-6}; the initial guess, right-hand side, and maximum number of iterations allowed were the same as for the previous set of the experiments. When ILU(0) preconditioning was used, no iterative solver

Table 5
Number of iterations for different orderings and preconditioners.

Preconditioner	P	Bi-CGSTAB				GMRES(20)				TFQMR			
		n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
ILU(1)	500	204	232	**157**	137	267	273	**256**	361	118	113	**106**	135
	1000	†	†	†	†	†	†	†	†	†	†	†	†
ILUT(10^{-2}, 5)	500	42	**13**	18	†	72	**24**	33	†	38	**14**	19	†
	1000	†	**24**	26	†	†	**48**	48	†	†	**25**	26	†
ILUT(10^{-3}, 10)	500	24	**8**	14	20	49	**14**	23	39	26	**8**	14	22
	1000	†	**12**	21	29	†	**23**	40	60	†	**13**	21	34

Table 6
Test problem information.

Matrix	N	NZ	Application	Source
watt2	1856	11550	Petroleum engineering	Harwell–Boeing
ale1590	1590	45090	Metal forming simulation	S. Barnard
kershaw60x60	10561	56257	Neutron diffusion	LANL
utm1700b	1700	21509	Plasma physics	SPARSKIT
utm3060	3060	42211	Plasma physics	SPARSKIT
utm5940	5940	83842	Plasma physics	SPARSKIT
fidap007	1633	54487	Incompressible flow	SPARSKIT

converged within the maximum allowed number of iterations, independent of the reordering used. However, with minimum degree the three solvers appeared to be slowly converging, whereas with the other reorderings the iteration either diverged or stagnated. The results for ILU(1) and ILUT preconditioning and various orderings are reported in Table 5. We note that the natural ordering and the Cuthill–McKee reorderings produced the same or comparable amount of fill-in for all preconditioners, whereas multiple minimum degree resulted in higher fill-in with ILU(1) and considerably less fill-in with the ILUT preconditioners with respect to the other reorderings. From these results, we observe that reorderings do not have a great impact on the performance of ILU(1) when $P = 500$. In contrast, reorderings make a difference when used with the ILUT preconditioners, with Cuthill–McKee producing the best results. Reverse Cuthill–McKee is much better than the natural ordering but is not quite as good as Cuthill–McKee. Multiple minimum degree is bad with ILUT$(10^{-2}, 5)$ but it performs well with ILUT$(10^{-3}, 10)$, although it is not as effective as Cuthill–McKee. Notice that for $P = 1000$, all preconditioners fail when the natural ordering is used. For this particular example Cuthill–McKee dramatically improves the performance of ILUT preconditioners. Finally, we mention that similar results were obtained with recirculating flow problems in which the coefficients of the first order terms in the convection-diffusion equation have variable sign.

3.2. Miscellaneous problems. The results in the previous subsection are relative to an important, but nevertheless rather special, class of problems. It is not clear to what extent, if any, those observations can be applied to other problems. For this reason, we discuss additional experiments performed on a selection of nonsymmetric matrices from various sources, including the Harwell–Boeing collection [19] and Saad's SPARSKIT [40]. These matrices arise from different application areas: oil reservoir modeling, plasma physics, neutron diffusion, metal forming simulation, etc. Some of these matrices arise from finite element modeling, and they have a much more complicated structure than those of the previous subsection. Also, they tend to be more ill conditioned. Some information about the matrices is provided in Table 6, where

Table 7
Number of iterations for different orderings, preconditioner ILU(0).

	Bi-CGSTAB				GMRES				TFQMR			
Matrix	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
watt2	36	41	**22**	36	76	96	**27**	49	56	71	**11**	16
ale1590	40	**25**	30	58	72	**46**	64	130	34	**25**	29	65
kershaw60x60	†	†	†	†	†	†	†	†	†	†	†	†
utm1700b	91	220	**54**	†	300	†	**166**	†	82	†	**57**	†
utm3060	131	133	**83**	†	†	†	**292**	†	108	185	**90**	†
utm5940	†	†	**159**	†	†	†	†	†	†	†	†	†
fidap007	†	†	†	†	†	†	†	†	†	†	†	†

Table 8
Number of iterations for different orderings, preconditioner ILU(1).

	Bi-CGSTAB				GMRES				TFQMR			
Matrix	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
watt2	28	31	**15**	17	17	17	**16**	17	†	†	**16**	16
ale1590	22	16	**16**	19	34	**26**	32	36	23	**16**	18	20
kershaw60x60	†	**83**	90	†	†	470	**441**	†	†	106	**106**	†
utm1700b	46	143	**33**	77	157	†	**82**	297	52	115	**35**	79
utm3060	61	108	**51**	154	217	†	**132**	†	60	116	**50**	157
utm5940	**171**	†	†	†	†	†	†	†	**221**	†	†	†
fidap007	†	†	†	†	†	†	†	†	†	†	†	†

N is the order of the matrix and NZ is the number of nonzeros.

The degree of difficulty of these problems varies from moderate (watt2) to extreme (utm5940 and fidap007). Concerning problem ale1590, which was provided by Barnard [2], the original ordering caused the coefficient matrix to have some zero entries on the main diagonal. This may cause trouble for the construction of ILU preconditioners. Therefore, the matrix was first reordered into a form with a zero-free diagonal, using a nonsymmetric permutation described in [20]. As for problem kershaw60x60, this matrix was extracted from the AUGUSTUS unstructured mesh diffusion package developed by Michael Hall at Los Alamos National Laboratory; see [32]. It should be mentioned that analogous results to those reported here for kershaw60x60 were obtained with different matrices extracted from this package. The convergence criterion used here is a residual norm reduction to less than 10^{-9}, due to the greater difficulty of these problems. All the remaining parameters are the same as those used in the previous subsection.

The ILUT parameters were $(10^{-2}, 5)$ and $(10^{-3}, 10)$ for the first four matrices (as in the experiments in Tables 3 and 4), whereas different parameters had to be used for the last three problems, due to their difficulty. For the matrix utm3060 the parameters used were $(10^{-3}, 10)$ and $(10^{-5}, 20)$; for utm5940, $(10^{-3}, 30)$ and $(10^{-4}, 40)$; and for fidap007, $(10^{-5}, 50)$ and $(10^{-7}, 70)$. In Tables 9 and 10 we refer to ILUT with these two sets of parameters as ILUT1 and ILUT2, respectively.

The minimum degree reordering always produced the least amount of fill-in in the preconditioner, while the original ordering and the Cuthill–McKee reorderings usually gave comparable fill-in. A notable exception is fidap007: for this matrix, the Cuthill–McKee reordering resulted in considerably more fill-in in the incomplete factors than the original ordering or reverse Cuthill–McKee. In Tables 9 and 10 the symbol * indicates that the maximum storage allowed for the preconditioner has been exceeded before the preconditioner construction was completed. This corresponds to approximately 300,000 nonzeros in the incomplete factors. In the tables, n/o refers to

TABLE 9
Number of iterations for different orderings, preconditioner ILUT1.

	Bi-CGSTAB				GMRES				TFQMR			
Matrix	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
watt2	31	27	**6**	13	99	80	**11**	16	32	32	**7**	†
ale1590	32	13	**11**	14	72	19	**19**	25	35	13	**13**	16
kershaw60x60	94	41	**35**	130	430	117	**103**	†	104	42	**39**	150
utm1700b	†	†	153	**125**	†	†	**459**	†	†	†	151	**152**
utm3060	173	199	**151**	172	†	†	†	†	189	204	**155**	229
utm5940	*	*	†	**217**	*	*	†	†	*	*	†	†
fidap007	†	†	†	†	†	†	†	†	†	†	†	†

TABLE 10
Number of iterations for different orderings, preconditioner ILUT2.

	Bi-CGSTAB				GMRES				TFQMR			
Matrix	n/o	CM	RC	MD	n/o	CM	RC	MD	n/o	CM	RC	MD
watt2	23	21	6	**7**	59	54	8	**11**	27	27	29	**7**
ale1590	30	11	**10**	13	51	**15**	16	20	28	11	**10**	16
kershaw60x60	45	22	**18**	35	165	55	**42**	113	49	23	**20**	37
utm1700b	†	198	**62**	92	†	†	**174**	293	†	210	**61**	100
utm3060	210	123	**73**	97	†	†	†	**293**	†	197	86	**96**
utm5940	*	*	*	**97**	*	*	*	†	*	*	*	**170**
fidap007	17	†	22	**18**	36	†	50	**36**	18	†	25	**21**

the original ordering of the matrices. With GMRES, the restart parameter used was $m = 20$ in all cases except for kershaw60x60, for which, due to the amount of storage required, $m = 14$ was used.

It should be mentioned that for the matrices from plasma physics, the standard implementation of the Cuthill–McKee reorderings caused a breakdown (zero pivot) of the ILU(1) and ILUT factorizations. Hence, it is possible for these reorderings to produce a poor pivot sequence. This difficulty was circumvented by applying to these matrices a slightly different version of (reverse) Cuthill–McKee, in which the first node is chosen as the initial node (that is, there is no search of a pseudoperipheral node), and no attempt is made to order nodes within a level set by increasing degree. Instead, the order of the nodes is determined by the order in which they are traversed; see [18]. With this implementation, no zero or very small pivots were encountered.

The results with these matrices are reported in Tables 7–10. They are somewhat less clear-cut than those for the convection-diffusion problems. Nevertheless, it can be seen that reorderings helped in a large majority of cases. While Cuthill–McKee and minimum degree did not perform well with ILU(0) and ILU(1), reverse Cuthill–McKee did with few exceptions. Reverse Cuthill–McKee was also useful with the ILUT preconditioners. For the more difficult problems which could be solved only allowing high amounts of fill-in in the factors, minimum degree proved useful.

We add that another version of ILUT, the ILUTP preconditioner (see [42]), was also tried. This is ILUT combined with a column pivoting strategy, and it is known to be sometimes better than ILUT, especially for problems leading to small pivots. For this set of experiments, however, ILUTP was found to be no better than ILUT.

3.3. Further analysis of the results. The results of the experiments presented show that a simple reordering of the coefficient matrix can bring about a dramatic improvement in the quality of incomplete factorization preconditioners. In particular, we saw problems where all preconditioned iterative solvers failed with the natural

TABLE 11
Norms of $A - \bar{L}\bar{U}$ and $I - A(\bar{L}\bar{U})^{-1}$ for different orderings, preconditioner $ILU(0)$.

ε^{-1}	n/o		CM		RC		MD	
	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$
100	1.46e-01	4.81e+01	1.49e-01	4.30e+01	1.46e-01	4.44e+01	9.29e-01	2.50e+01
200	2.70e-01	1.09e+04	2.71e-01	9.74e+03	2.70e-01	1.10e+04	1.36e+00	4.89e+01
300	3.23e-01	8.89e+04	3.26e-01	8.15e+04	3.23e-01	9.08e+04	1.88e+00	7.42e+01
400	3.53e-01	2.68e+05	3.57e-01	2.73e+05	3.53e-01	2.74e+05	2.41e+00	9.97e+01
500	3.72e-01	5.28e+05	3.78e-01	1.01e+06	3.72e-01	5.40e+05	2.94e+00	1.25e+02
600	3.86e-01	8.45e+05	3.92e-01	1.01e+07	3.86e-01	8.61e+05	3.48e+00	1.50e+02
700	3.97e-01	1.34e+06	4.04e-01	8.60e+07	3.97e-01	1.30e+06	4.01e+00	1.75e+02
800	4.06e-01	3.26e+06	4.13e-01	4.86e+08	4.06e-01	2.79e+06	4.53e+00	2.00e+02
900	4.14e-01	9.93e+06	4.20e-01	1.98e+09	4.14e-01	8.25e+06	5.05e+00	2.25e+02
1000	4.22e-01	2.69e+07	4.26e-01	6.25e+09	4.22e-01	2.26e+07	5.57e+00	2.50e+02

TABLE 12
Norms of $A - \bar{L}\bar{U}$ and $I - A(\bar{L}\bar{U})^{-1}$ for different orderings, preconditioner $ILU(1)$.

ε^{-1}	n/o		CM		RC		MD	
	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$
100	4.60e-02	1.70e+00	4.90e-02	2.13e+00	4.59e-02	2.09e+00	4.21e-01	1.30e+01
200	1.31e-01	6.56e+00	8.57e-02	4.68e+00	7.94e-02	5.21e+00	5.49e-01	1.89e+01
300	1.79e-01	1.08e+01	1.09e-01	6.47e+00	9.56e-02	7.51e+00	6.95e-01	2.53e+01
400	2.09e-01	1.44e+01	1.30e-01	8.12e+00	1.06e-01	9.42e+00	8.49e-01	3.23e+01
500	2.29e-01	1.77e+01	1.50e-01	9.77e+00	1.12e-01	1.11e+01	1.01e+00	3.96e+01
600	2.43e-01	2.08e+01	1.69e-01	1.15e+01	1.18e-01	1.27e+01	1.17e+00	4.71e+01
700	2.55e-01	2.36e+01	1.86e-01	1.32e+01	1.22e-01	1.41e+01	1.33e+00	5.46e+01
800	2.64e-01	2.63e+01	2.02e-01	1.49e+01	1.26e-01	1.56e+01	1.49e+00	6.21e+01
900	2.71e-01	2.89e+01	2.17e-01	1.67e+01	1.29e-01	1.70e+01	1.66e+00	6.96e+01
1000	2.77e-01	3.14e+01	2.30e-01	1.85e+01	1.33e-01	1.83e+01	1.82e+00	7.71e+01

ordering and all converged rapidly after a symmetric permutation of the coefficient matrix. In this subsection, we investigate the reasons behind these observations.

An incomplete factorization preconditioner can fail or behave poorly for several reasons. A common cause of failure is instability of the incomplete factorization, which is caused by numerically zero pivots or exceedingly small ones. The result of this type of instability is that the incomplete factorization is very inaccurate, that is, the norm of the residual matrix $R = A - \bar{L}\bar{U}$ is large. This is a very real possibility for matrices that do not have some form of diagonal dominance and for highly unstructured problems. Of course, an inaccurate factorization can also occur in the absence of small pivots, when many large fill-ins are dropped from the incomplete factors. Another kind of instability, which can take place whether or not small pivots occur, is severe ill-conditioning of the triangular factors, which reflects the instability of the long recurrences involved in the forward and backward solves when the preconditioning is applied [25], [47]. In this situation, $\|R\|_F$ need not be very large, but $\|I - A(\bar{L}\bar{U})^{-1}\|_F = \|R(\bar{L}\bar{U})^{-1}\|_F$ will be. Again, this is a common situation when the coefficient matrix is far from being diagonally dominant. Of course, both types of instabilities can simultaneously occur for a given problem; see [11] for an extensive experimental study of the causes of failure of incomplete factorizations.

In order to gain some insight about the effect of reorderings, we computed the Frobenius norms of R and $R(\bar{L}\bar{U})^{-1}$ for each test matrix, reordering, and preconditioner. Those for the matrices arising from the discretization of problem (1) are reported in Tables 11–14, where $N1 = \|A - \bar{L}\bar{U}\|_F$ and $N2 = \|I - A(\bar{L}\bar{U})^{-1}\|_F$. Loosely speaking, $N1$ measures the accuracy of the incomplete factorization, whereas

TABLE 13
Norms of $A - \bar{L}\bar{U}$ *and* $I - A(\bar{L}\bar{U})^{-1}$ *for different orderings, preconditioner* $ILUT(10^{-2}, 5)$.

	n/o		CM		RC		MD	
ε^{-1}	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$
100	9.20e-03	5.65e-01	4.96e-03	2.06e-01	2.57e-03	1.07e-01	1.39e-01	4.64e+00
200	3.09e-02	2.64e+00	5.22e-03	2.82e-01	4.71e-03	2.48e-01	3.13e-01	1.01e+01
300	5.64e-02	6.21e+00	1.13e-02	7.42e-01	8.58e-03	4.93e-01	5.08e-01	1.83e+01
400	7.75e-02	9.22e+00	1.91e-02	1.31e+00	1.29e-02	8.31e-01	7.09e-01	3.45e+01
500	1.00e-01	1.21e+01	2.81e-02	1.98e+00	1.68e-02	1.21e+00	9.98e-01	1.62e+02
600	1.18e-01	1.45e+01	3.82e-02	2.78e+00	2.04e-02	1.62e+00	1.45e+00	2.58e+03
700	1.43e-01	1.84e+01	5.20e-02	3.96e+00	2.43e-02	2.07e+00	2.03e+00	1.69e+03
800	1.78e-01	2.79e+01	6.21e-02	4.89e+00	2.83e-02	2.54e+00	7.39e+01	5.81e+06
900	6.57e+20	1.69e+33	9.88e-02	8.86e+00	3.22e-02	3.08e+00	9.37e+00	1.12e+05
1000	6.47e+28	3.98e+43	9.82e-02	8.33e+00	3.54e-02	3.52e+00	2.28e+01	3.71e+06

TABLE 14
Norms of $A - \bar{L}\bar{U}$ *and* $I - A(\bar{L}\bar{U})^{-1}$ *for different orderings, preconditioner* $ILUT(10^{-3}, 10)$.

	n/o		CM		RC		MD	
ε^{-1}	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$	$N1$	$N2$
100	1.48e-03	8.56e-02	5.09e-04	2.05e-02	3.77e-04	1.63e-02	4.76e-02	1.62e+00
200	7.18e-03	4.99e-01	6.05e-04	2.93e-02	4.97e-04	2.68e-02	1.13e-01	3.85e+00
300	1.82e-02	1.36e+00	1.51e-03	9.48e-02	1.11e-03	6.76e-02	1.79e-01	6.24e+00
400	3.14e-02	2.49e+00	3.41e-03	2.36e-01	2.06e-03	1.39e-01	2.55e-01	9.00e+00
500	4.61e-02	3.78e+00	6.48e-03	4.78e-01	3.08e-03	2.28e-01	3.39e-01	1.21e+01
600	6.16e-02	5.42e+00	1.02e-02	7.86e-01	4.26e-03	3.41e-01	4.46e-01	1.55e+01
700	8.37e-02	8.56e+00	1.45e-02	1.15e+00	5.87e-03	5.01e-01	5.48e-01	1.92e+01
800	9.85e-02	1.08e+01	1.90e-02	1.55e+00	7.88e-03	6.98e-01	6.45e-01	2.29e+01
900	1.19e-01	1.43e+01	2.45e-02	2.08e+00	1.00e-02	9.34e-01	7.42e-01	2.68e+01
1000	1.48e-01	1.98e+01	2.96e-02	2.54e+00	1.18e-02	1.13e+00	8.39e-01	3.08e+01

$N2$ measures its stability (in the sense of [25]). We also monitored the size of the pivots in the course of the incomplete factorizations, and we did not find any very small pivots. Hence, failure or poor behavior of an incomplete factorization could be due to significantly large fill-ins having been dropped, to unstable triangular solves, or both.

The results in Table 11 give a clear explanation of the convergence behavior of iterative methods with ILU(0) preconditioning reported in Table 1. For the natural, Cuthill–McKee, and reverse Cuthill–McKee orderings the degradation and eventual failure in the convergence as ε^{-1} increases is not due to inaccuracy of the incomplete factorizations, but to instability of the triangular solves. As ε^{-1} increases, the condition number of the no-fill incomplete factors grows rapidly, the preconditioned matrix $A(\bar{L}\bar{U})^{-1}$ becomes more and more ill conditioned, and the number of iterations increases. Furthermore, for large enough ε^{-1}, Elman [25] observed that the symmetric part of $A(\bar{L}\bar{U})^{-1}$ becomes indefinite, and this in turn can cause failure of the Krylov subspace accelerators. Inspection of the last column in Table 11 reveals that minimum degree has the effect of stabilizing the ILU(0) triangular factors. The preconditioner remains well conditioned even for large values of ε^{-1}, and all three Krylov subspace methods converge. The fact that the number of iterations still increases with increasing ε^{-1} appears to be due to the fact that the ILU(0) factorization becomes less accurate, as measured by $N1$. We observe that for the moderately nonsymmetric problems ($\varepsilon^{-1} \leq 400$) the number of iterations is almost directly related to $N1 = \|A - \bar{L}\bar{U}\|_F = \|R\|_F$, whereas for $\varepsilon^{-1} \geq 500$ this norm alone is not a good indicator of the effectiveness of the preconditioner and $N2 = \|I - A(\bar{L}\bar{U})^{-1}\|_F$ becomes a more reliable indicator. Notice that minimum degree always results in less accurate ILU(0) factors (larger $N1$) than the other orderings.

The results in Table 2 show that ILU(1) preconditioning is more robust and effective than ILU(0) for this class of problems, and the Frobenius norms presented in Table 12 show that this is due to the fact that ILU(1) does not suffer from the kind of instability that plagues ILU(0). Because of this, $N1 = \|A - \bar{L}\bar{U}\|_F$ is now a fairly accurate indicator of the performance of the preconditioner for all values of ε^{-1}. The values reported for $N1$ indicate that the Cuthill–McKee reorderings outperform the natural ordering because they result in more accurate ILU(1) factors and that minimum degree is inferior to the other orderings because the incomplete factorization is less accurate.

We mention that some ad hoc stabilization techniques to be used with ILU(0) for convection-diffusion problems have been proposed in [26] and [47]. Our results indicate that using ILU(1) with a level set reordering of the matrix offers a simple solution to the instability problem.

Similar results apply to the ILUT preconditioners (Tables 13 and 14). However, there are two phenomena that occur with ILUT$(10^{-2}, 5)$ and not with ILUT$(10^{-3}, 10)$ that deserve to be mentioned. For $\varepsilon^{-1} \geq 900$, the ILUT$(10^{-2}, 5)$ preconditioner with natural ordering fails rather dramatically. An inspection of the corresponding entries of Table 13 shows that this is due to the simultaneous occurrence of inaccuracy in the factorization (large $\|R\|_F$—too many large fill-ins have been dropped) *and* instability in the triangular factors (as revealed by a much larger value of $\|R(\bar{L}\bar{U})^{-1}\|_F$). This is an illustration of the fact that for strongly nonsymmetric problems, increasing fill-in in the incomplete factors does not necessarily result in an improved preconditioner, unless the factorization approaches an exact one; see also [11]. We mention that the number of nonzeros in the factors is considerably higher for ILUT$(10^{-2}, 5)$ than for ILU(1). The other interesting phenomenon is that with ILUT$(10^{-2}, 5)$, the incomplete factors obtained with minimum degree are not only less accurate than those for the other reorderings but also unstable when ε^{-1} becomes large. This is the opposite of what happens for ILU(0) and confirms that the relative performance of a given reordering is different for different preconditioning strategies, as already observed in [21]. When ILUT$(10^{-3}, 10)$ is used, the accuracy of the incomplete factorization approaches that of a direct solve and none of the orderings suffers from instability (note that the original coefficient matrix A is fairly well conditioned for all values of ε^{-1}). The norm of $R = A - \bar{L}\bar{U}$ again becomes a very reliable indicator of the performance of the preconditioners corresponding to the different permutations. The Cuthill–McKee reorderings give better results than the other orderings because they make the incomplete factorization more accurate, as is indicated by the values for $N1$ reported in Table 14. For fixed values of the ILUT parameters, the amount of fill-in in the incomplete factors is only slightly less with the Cuthill–McKee reorderings than with the natural ordering, whereas the number of nonzeros in the *complete* LU factors is much less. Hence, when compared to the natural ordering, the Cuthill–McKee reorderings allow one to compute a more accurate incomplete factorization for roughly the same arithmetic and storage costs for this class of problems.

The Frobenius norms $\|R\|_F$ and $\|R(\bar{L}\bar{U})^{-1}\|_F$ were also computed for the two matrices arising from problem (2). For ILU(0), the failures with the natural ordering and the Cuthill–McKee reorderings were due to the concurrent effect of inaccuracy and instability of the triangular solves (again, no small pivots arise for these problems). The instability was especially severe for the $P = 1000$ case. For instance, with the

Cuthill–McKee ordering we found $N1 = 9.77\text{e}{+}07$ and $N2 = 1.03\text{e}{+}16$. On the other hand, no instabilities occurred with minimum degree, and the failures with this reordering were due to inaccurate factorizations ($N1 = 1.94\text{e}{+}08$ for $P = 1000$). With minimum degree, the computed residual was reduced to about 10^{-5} by all three iterative methods preconditioned with ILU(0) when the maximum number of iterations was reached. With the natural ordering and Cuthill–McKee reorderings, on the other hand, there was divergence or stagnation at much higher values of the residual. Thus, it appears that instability in the preconditioner has a more devastating effect than low accuracy of the factorization.

With ILU(1), no instabilities occurred. The failures with the natural ordering and with multiple minimum degree for $P = 1000$ were due to low accuracy of the factorization (large $N1$). With the ILUT preconditioners, the failures with the natural ordering are due to the simultaneous occurrence of inaccuracy in the factorization and unstable triangular solves, very much like the case in Table 13 for $\varepsilon^{-1} \geq 900$. All the other orderings produced stable incomplete factorizations. The failures with the minimum degree ordering and ILUT$(10^{-2}, 5)$ preconditioning were due to inaccuracy of the factorization.

Again, the best results are obtained with the Cuthill–McKee reorderings and ILUT preconditioning, which yield accurate and stable factorizations. We note that for these problems, Cuthill–McKee is somewhat better than reverse Cuthill–McKee.

The Frobenius norms of R and $R(\bar{L}\bar{U})^{-1}$ were also computed for the problems of section 3.2. In most cases, when considered together they were found to give a qualitative explanation of the observed convergence behavior. Whenever a preconditioner failed, it was usually due to inaccuracy rather than instability, with the exception of fidap007 with ILUT preconditioning and Cuthill–McKee reordering, for which the factorization was both inaccurate and severely unstable.

4. Conclusions. We have provided evidence that reorderings originally designed for use with sparse direct solvers can significantly improve the performance of iterative methods preconditioned with incomplete LU factorizations. While not entirely new, an examination of the literature reveals that this fact is not widely known.

The benefit of reordering the coefficient matrix depends in part on how far the matrix is from being symmetric and diagonally dominant, as well as on the type of incomplete factorization preconditioner used. In our experiments with regular grid problems, we found that when the coefficient matrix is nearly symmetric, very little is gained from reordering it. On the other hand, if the matrix is strongly nonsymmetric, large reductions of the number of iterations can be obtained by (symmetrically) reordering the matrix.

A somewhat surprising result of our experiments is that the “natural” or “original” ordering of the test matrices used in this study is almost never the best from the point of view of incomplete factorization preconditioning and is very often the worst. More specifically, the original ordering was found to give the best results in only 13 cases out of the 228 comparisons reported in this paper. Reverse Cuthill–McKee gave the best results in 132 cases, Cuthill–McKee in 57 cases, and multiple minimum degree in 30 cases. The original ordering was found to be worse than the other orderings also from the point of view of robustness: there were 61 failures with the original ordering, 54 with multiple minimum degree, 48 with Cuthill–McKee, and only 37 with reverse Cuthill–McKee. There were 26 cases where the original ordering led to a failure and reverse Cuthill–McKee succeeded, but only two cases where reverse Cuthill–McKee failed and the original ordering succeeded (matrix utm5940 with ILU(1) preconditioning; see Table 10).

It should be stressed that in most cases for which reverse Cuthill–McKee was not best, it still gave good results (that is, it was not found to be much worse than the best ordering). Hence, overall, reverse Cuthill–McKee appears to be superior to the other orderings in the context of incomplete factorization preconditioning. As revealed by a direct inspection of the residual matrices $A - \bar{L}\bar{U}$, in most cases this was simply due to the fact that this reordering produced more accurate (as measured by $\|A - \bar{L}\bar{U}\|_F$) incomplete factorizations than those obtained with the natural ordering, with a comparable amount of fill-in in the factors. In some cases the improvement was due to the fact that the reordering resulted in a stabilization of the incomplete triangular factors (as measured by $\|I - A(\bar{L}\bar{U})^{-1}\|_F$). However, none of the orderings considered in this paper were found to be completely immune from potential instabilities in the corresponding triangular solves. In most cases where instabilities occurred, the problem disappeared by allowing more fill-in in the incomplete factors, but not always. Indeed, there were a few cases where increasing the amount of fill-in made things worse in the sense that the instability of the factors increased; see also [11].

In general, Cuthill–McKee cannot be recommended. While it performed well on many problems, its behavior is rather erratic. Reverse Cuthill–McKee should be preferred.

The minimum degree reordering was found to be inferior to the level set reorderings in general, but often better than the original ordering. It is well known that for the purpose of *complete* sparse factorization, minimum degree is usually much more effective at preserving sparsity than level set reorderings. Thus, this reordering could be useful when the LU factorization with the original ordering suffers from extremely high fill-in, and a sparse preconditioner is sought. For the same choice of the ILUT parameters, this reordering always resulted in incomplete factors which were considerably more sparse than those obtained with the other reorderings. While it is true that minimum degree is more expensive to compute than the level set reorderings, this cost is usually of the order of only a few iterations.

Of course, reverse Cuthill–McKee is not trouble-free. The quality of the corresponding preconditioner will be affected, in general, by the choice of the initial node and by the ordering of nodes within level sets. In particular, it is not clear how different tie-breaking strategies will affect the incomplete factorization. It is also possible that some orderings within level sets will produce a poor pivot sequence and a breakdown of the incomplete factorization process. Moreover, it is easy to contrive examples where reverse Cuthill–McKee will be a poor ordering, for example, by constructing a convection-diffusion problem for which the reverse Cuthill–McKee ordering of the grid points goes against the flow direction.

Nevertheless, based on the results of our experiments, we conclude that much can be gained from reordering strongly nonsymmetric matrices before performing an incomplete factorization and not much should be lost, particularly when the reverse Cuthill–McKee reordering is used. For convection-diffusion problems on rectangular grids, ILU(1) or ILUT preconditioning combined with reverse Cuthill–McKee is recommended, whereas the lexicographic ordering behaves rather poorly and should be avoided. For matrices that do not have a "natural" ordering, such as those arising from unstructured meshes, we recommend reverse Cuthill–McKee as the original ordering. A similar conclusion was reached in [21] for symmetric matrices arising from the finite element method.

Concerning possible developments of this study, an interesting possibility would be to consider a red-black approach, where the reduced system is reordered with reverse Cuthill–McKee. Some promising results with this approach were reported in [5]. It would also be interesting to study the effects of combining nonsymmetric permutations designed for moving large entries to the diagonal (see [20]) with the symmetric permutations considered in this paper.

Finally, there are some open questions which warrant further investigation. As already mentioned, some understanding of the effect of the choice of the initial node and of the ordering within level sets on the performance of (reverse) Cuthill–McKee would be welcome. Also, with reference to the linear systems arising from the discretization of model problem (1) or similar ones, it would be desirable to understand why the ILU(0) factors computed with the minimum degree ordering do not suffer from the instability that occurs when the natural ordering (or the equivalent level set orderings) are used. Likewise, it would be instructive to understand why the ILU(1) factors were found to be stable regardless of the ordering used to compute them; see Table 2. At present, we are unable to see how Elman's analysis [25] for the ILU(0) preconditioner with the natural and equivalent orderings could be applied to more complicated preconditioners and to other orderings.

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