Numer. Math. (2006) 103: 173–196
DOI 10.1007/s00211-006-0679-9

Numerische
Mathematik

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On the eigenvalues of a class of saddle point matrices

Received: 1 June 2005 / Revised: 6 December 2005 / Published online: 21 March 2006

Abstract We study spectral properties of a class of block 2×2 matrices that arise in the solution of saddle point problems. These matrices are obtained by a sign change in the second block equation of the symmetric saddle point linear system. We give conditions for having a (positive) real spectrum and for ensuring diagonalizability of the matrix. In particular, we show that these properties hold for the discrete Stokes operator, and we discuss the implications of our characterization for augmented Lagrangian formulations, for Krylov subspace solvers and for certain types of preconditioners.

Mathematics Subject Classifications: Primary 65F10 · 65N22 · 65F50 · Secondary 76D07

1 Introduction

In this paper we investigate spectral properties of block 2×2 matrices of the form

The work of this author was supported in part by the National Science Foundation grant DMS-0207599.
Revision dated 5 December 2005.

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$$M_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite (SPD), $B \in \mathbb{R}^{m\times n}$ with $m \le n$, and $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite (possibly $C = O$). If we define

$$M_+ = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} I_n & O \\ O & -I_m \end{bmatrix},$$

we have $M_- = JM_+$. Note that M_+ is symmetric, whereas M_- is nonsymmetric; however,

$$M_- J = J M_-^T, \tag{1.2}$$

i.e., M_- is J-symmetric. Consider the block factorization

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} = \begin{bmatrix} I_n & O \\ BA^{-1} & I_m \end{bmatrix} \begin{bmatrix} A & O \\ O & -S \end{bmatrix} \begin{bmatrix} I_n & A^{-1}B^T \\ O & I_m \end{bmatrix},$$

where $S = C + BA^{-1}B^T$ is the Schur complement of A in M_-. It follows that if $\ker(C) \cap \ker(B^T) = \{0\}$ (i.e., if C is positive definite on $\ker(B^T)$), the matrix M_+ is indefinite with exactly n positive and m negative eigenvalues. In contrast, the nonsymmetric matrix M_- is positive semidefinite, i.e., the symmetric part of M_-,

$$\frac{1}{2}(M_- + M_-^T) = \begin{bmatrix} A & O \\ O & C \end{bmatrix},$$

is positive semidefinite. Together with the assumption that $\ker(C) \cap \ker(B^T) = \{0\}$, this in turn implies that all the eigenvalues of M_- have positive real part [6,28]; thus, M_- is a *positive stable* matrix.

Matrices of the form (1.1) frequently occur in the solution of saddle point problems; these include e.g. mixed finite element formulations of elliptic PDEs, fluid dynamics and constrained optimization problems, and so forth; see [7] for an extensive survey. We will also consider the case of A symmetric positive semidefinite, which arises in many applications. Although the symmetric form M_+ is generally preferred, there are situations where it is of interest to consider the nonsymmetric form M_- instead. One such reason is that the nonsymmetric form permits the use of special preconditioners which are not natural for the symmetric form; see [5,6]. Another reason, which is the main motivation for our present work, is that in some cases the eigenvalues of M_- turn out to be all real (and, of course, positive). When M_- is diagonalizable, this implies the existence of a nonstandard inner product on $\mathbb{R}^{n+m}$ with respect to which M_- is symmetric positive definite. As we shall see, these properties may always be obtained, at least in principle, using appropriate scalings or by means of augmented Lagrangian techniques. Our theory also provides a general framework for the analysis of certain preconditioners for saddle point problems.

2 Analysis of the eigenvalues

We consider the problem

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \text{or } M_- q = \lambda q. \tag{2.1}$$

We make the general assumption that A and C are symmetric. Later we will also require that A be positive definite and C be positive semidefinite. The spectral properties of the problem above have been studied in [12] for $A = \eta I_n$ ($\eta > 0$) and $C = O$ and in [28] for $C = O$. The results given here are more general and complete; for instance, our conditions for the reality of the spectrum do not appear to have been given before. In the sequel we will consider the following two block equations, equivalent to (2.1),

$$Au + B^T v = \lambda u \tag{2.2}$$

$$-Bu + Cv = \lambda v. \tag{2.3}$$

Note that the problem above can be equivalently written as

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} I_n & O \\ O & -I_m \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \text{or } M_+ q = \lambda J q, \tag{2.4}$$

a generalized eigenproblem for the symmetric pencil (M_+, J). In the following, we will say that two vectors $x, y \in \mathbb{C}^{n+m}$ are J-orthogonal if $x^* J y = 0$, i.e., if they are orthogonal with respect to the indefinite inner product associated with J. Here and elsewhere in the paper, x^* denotes the conjugate transpose of vector x.

2.1 General results on the eigenvalues of M_-

In this section we give some general conditions for the reality of the spectrum of M_-, and on the number of non-real eigenvalues, if they exist. We first recall a key result for our analysis. This result is also important for an understanding of the eigenvector structure of M_-. Recall that an eigenspace of a matrix is *simple* if its eigenvalues are, counting multiplicities, disjoint from the other eigenvalues of the matrix; see [31, p. 244].

Theorem 2.1 ([17, Thm. 2.5]) *Let λ, η be eigenvalues of M_- such that $\lambda \neq \bar{\eta}$. Then the simple invariant subspace associated with λ is J-orthogonal to the simple invariant subspace of η.*

The condition $\lambda \neq \bar{\eta}$ is satisfied by all complex eigenvalues with non-zero imaginary parts that are not conjugate of each other. Assuming that M_- is diagonalizable, the theorem above implies that the eigenvectors of real distinct eigenvalues and of non-real, non-conjugate eigenvalues are all mutually J-orthogonal. Note that the theorem above does not require M_- to be diagonalizable. However, here we make the assumption that M_- is diagonalizable, in order to simplify our analysis. This assumptions is usually satisfied in the applications we are interested in. See also the discussion at the end of this subsection.

Denoting $[u; v]$ an eigenvector corresponding to an eigenvalue with non-zero imaginary part, it can be easily seen that

$$[u^*, v^*]J \begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

that is, $[u; v]$ is J-neutral (or J-isotropic). This property is equivalent to the following equality, which is important for our analysis:

$$\|u\| = \|v\|. \tag{2.5}$$

(Here and throughout the paper, $\| \cdot \|$ denotes the Euclidean norm for vectors, and the induced norm for matrices.) The diagonalizability assumption ensures that eigenvectors corresponding to multiple real eigenvalues are not J-neutral (*cf.* [17, p. 36]).

We start with a simple but significant result, which follows from letting $v = 0$ and then $u = 0$ in (2.2) and (2.3).

Proposition 2.2 *The matrix M_- has at most $n - m$ real eigenvalues satisfying $Au = \lambda u$, $Bu = 0$, with corresponding eigenvectors* $[u; 0]$. *Furthermore, if B^T is rank deficient with null space of dimension r, then M_- has at most r real eigenvalues satisfying $Cv = \lambda v$, $B^T v = 0$, with corresponding eigenvectors* $[0; v]$.

We can now state a result on the number of non-real eigenvalues.

Proposition 2.3 *Let A, C be symmetric. Then the matrix in (2.4) has at most* $2m$ *eigenvalues with non-zero imaginary part, counting conjugates.*

Proof Let $[u_i; v_i], i = 1, \dots, k, u_i \in \mathbb{C}^n, v_i \in \mathbb{C}^m$, be the eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_k$ with non-zero imaginary part, such that $\lambda_i \neq \bar{\lambda}_j$ (that is, no complex conjugates are included). We recall that $[u_i; v_i]^* J [u_j; v_j] = 0, \forall i, j = 1, \dots k$. Let $U = [u_1, \dots, u_k]$, $V = [v_1, \dots, v_k]$. Then $U^*U - V^*V = 0$, that is $U^*U = V^*V$.

We prove the assertion by contradiction. Assume that $k > m$. Let $V = Q_v R$ be a reduced QR factorization of V, with $Q_v \in \mathbb{C}^{m\times\ell}$, $R \in \mathbb{C}^{\ell\times k}$, $\ell \leq m$. Then $V^*V = R^*R = U^*U$. Therefore, a reduced QR factorization of U is given by $U = Q_u R$ with some $Q_u \in \mathbb{C}^{n\times\ell}$ having orthonormal columns. Hence

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} Q_u \\ Q_v \end{bmatrix} R \qquad \text{of rank} \;\; \ell \leq m < k.$$

This is a contradiction, since the matrix $[U; V]$ of eigenvectors is supposed to have full rank equal to k. Therefore, it must be $k \leq m$, that is, there are at most m non-conjugate eigenvalues with non-zero imaginary part. The same proof can be carried out with the remaining (conjugate) eigenpairs, completing the proof. □

The previous result can also be obtained as a consequence of Corollary S5.2 in [16, p. 378]. Our proof, however, is more self-contained.

The following proposition states necessary and sufficient conditions for β to be an eigenvalue of M_- in the practically important case $C = \beta I_m$; see, *e.g.*, [3, p. 398]. Its proof is straighthforward and can be omitted.

Proposition 2.4 *Assume A is symmetric, B^T has full column rank, and $C = \beta I_m$, with $\beta \in \mathbb{R}$.*

1. *If β is an eigenvalue of M_- with corresponding eigenvector $x = [u; v]$, then $u \in \ker(B)$, and β is an eigenvalue of A associated with u if and only if $v = 0$.*
2. *Let $(\lambda, [u; v])$ be an eigenpair of M_-, $u \neq 0$. Then β is an eigenvalue of A associated with u if and only if $\lambda = \beta$, in which case it is necessarily $v = 0$ and $u \in \ker(B)$.*

Next, we give our main result of this section: conditions for having only real eigenvalues in the case when $C = \beta I_m$, with $\beta \geq 0$.

Proposition 2.5 *Assume $C = \beta I_m$ with $\beta \geq 0$. Let $(\lambda, [u; v])$ be an eigenpair of M_-. Then either $\lambda = \beta$, or λ is real if and only if*

$$\left(\frac{u^*(A + \beta I_n)u}{u^*u}\right)^2 \geq 4\frac{u^*(B^T B + \beta A)u}{u^*u}. \tag{2.6}$$

Proof If $\lambda = \beta > 0$ then λ is real. For $C = \beta I_m$ and $\lambda \neq \beta$, we use the equation in (2.3) to get $v = -Bu/(\lambda - \beta)$, and substitute v into (2.2). After rearranging terms, we obtain $\lambda^2 u - \lambda(A + \beta I_n)u + (B^T B + \beta A)u = 0$.

For $u \neq 0$, multiplying from the left by u^* yields the following quadratic equation in λ with real coefficients,

$$\lambda^2 u^*u - \lambda u^*(A + \beta I_n)u + u^*(B^T B + \beta A)u = 0.$$

The two solutions are given by

$$\lambda = \frac{1}{2}\frac{u^*(A + \beta I_n)u}{u^*u} \pm \frac{1}{2}\sqrt{\left(\frac{u^*(A + \beta I_n)u}{u^*u}\right)^2 - 4\frac{u^*(B^T B + \beta A)u}{u^*u}}.$$

Therefore, the two roots are real if and only if (2.6) holds. □

The condition in Proposition 2.5 can be rewritten as

$$\frac{u^*(A + \beta I_n)u}{u^*u} \geq 4\,\frac{u^*(B^T B + \beta A)u}{u^*(A + \beta I_n)u}.$$

Since $u^*(A + \beta I_n)u \geq u^*Au$, we have

$$\frac{u^*(B^T B + \beta A)u}{u^*(A + \beta I_n)u} \leq \frac{u^*(B^T B + \beta A)u}{u^*Au} = \frac{q^*(A^{-\frac{1}{2}} B^T B A^{-\frac{1}{2}} + \beta I_n)q}{q^*q},$$

$$q = A^{\frac{1}{2}}u.$$

Therefore, if $\lambda_{\min}(A + \beta I_n) \geq 4\lambda_{\max}(BA^{-1}B^T + \beta I_m)$, the condition of Proposition 2.5 is satisfied, since

$$\begin{aligned}\frac{u^*(A + \beta I_n)u}{u^*u} &\geq \lambda_{\min}(A + \beta I_n) \geq 4\,\lambda_{\max}(BA^{-1}B^T + \beta I_m)\\ &\geq 4\,\frac{q^*(A^{-\frac{1}{2}} B^T B A^{-\frac{1}{2}} + \beta I_n)q}{q^*q} \geq 4\,\frac{u^*(B^T B + \beta A)u}{u^*(A + \beta I_n)u}.\end{aligned}$$

We have thus proved the following result.

Corollary 2.6 *Assume the same notation as in Proposition (2.5) holds. If* $\lambda_{\min}(A+\beta I_n) \geq 4\,\lambda_{\max}(BA^{-1}B^T + \beta I_m)$*, then all eigenvalues of* M_- *are real.*

The reality condition in the foregoing corollary can be restated as $\lambda_{\min}(A) \geq 4\,\lambda_{\max}(BA^{-1}B^T) + 3\,\beta$; for $\beta = 0$, it can also be expressed as $\lambda_{\min}(A) \geq 4\,\|S\|$.

We also note that the condition is sufficient, but not necessary. For instance, the matrix

$$M_- = \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 1 \\ \hline 0 & -1 & 0 \end{array}\right] \tag{2.7}$$

has real spectrum but does not satisfy the condition.

The results of Proposition 2.5 and of Corollary 2.6 can be generalized to matrices of the form

$$\widehat{M} = \begin{bmatrix} A & B_1^T \\ -B_2 & C \end{bmatrix}, \tag{2.8}$$

under certain conditions on B_1, B_2.

Corollary 2.7 *Let* $A = A^T \in \mathbb{R}^{n\times n}$ *be positive definite,* $B_1, B_2 \in \mathbb{R}^{m\times n}$ *be such that* $B_1^T B_2$ *is symmetric, and let* $C = \beta I_m$, $\beta \geq 0$*. Let* $(\lambda, [u; v])$ *be an eigenpair of* $\widehat{M}$*. Then* λ *is real if and only if*

$$\left(\frac{u^*(A+\beta I_n)u}{u^*u}\right)^2 \geq 4\,\frac{u^*(B_1^T B_2 + \beta A)u}{u^*u}.$$

Moreover, all eigenvalues are real if $\lambda_{\min}(A+\beta I_n) \geq 4\,\lambda_{\max}(B_2 A^{-1} B_1^T + \beta I_m)$.

Proof The proof proceeds as for Proposition 2.3 and for Corollary 2.6. Note that the hypothesis that $B_1^T B_2$ is symmetric implies that the eigenvalues of $A^{-1}B_1^T B_2$ are real, and therefore (applying Theorem 1.3.20 in [19]) the eigenvalues of $B_2 A^{-1} B_1^T$ are also real. □

This class of matrices includes the case where $B_1 = B$ and $B_2 = \tau B$, with $\tau > 0$. Note that for $\tau < 0$ the matrix $\widehat{M}$ would be the product of two symmetric matrices, one of which (diagonal) positive definite, so that all eigenvalues would be real. As a special case consider the parametrized family of matrices

$$M(\tau) = \begin{bmatrix} A & B^T \\ -\tau B & \tau\beta I_m \end{bmatrix}.$$

Assuming A is SPD, we can define

$$\tau_* \equiv \frac{1}{4}\,\frac{\lambda_{\min}(A)}{\lambda_{\max}(BA^{-1}B^T + \beta I_m)}.$$

Then $\tau_* > 0$, and $M(\tau)$ has all the eigenvalues real for $\tau \in (-\infty, \tau_*]$. Moreover, the symmetrically scaled matrix

$$\begin{bmatrix} A & \sqrt{\tau}B^T \\ -\sqrt{\tau}B & \tau\beta I_m \end{bmatrix}$$

has all the eigenvalues positive and real for $0 < \tau \le \tau_*$.

Our sufficient condition for the reality of the spectrum can be further extended to generalized eigenvalue problems of the type

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} K & O \\ O & -H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{2.9}$$

where K, H are real symmetric and positive definite. This will be useful when we discuss applications to preconditioning.

Remark 2.8 For A semidefinite and singular, non-real eigenvalues should be expected regardless of B (as long as $B \neq O$). For example, the matrix

$$M_- = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & \varepsilon \\ \hline 0 & -\varepsilon & 0 \end{array}\right]$$

has non-real eigenvalues for all $\varepsilon \neq 0$. The situation may be different if C is allowed to be nonzero. For instance, the spectrum of the matrix

$$M_- = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & \varepsilon \\ \hline 0 & -\varepsilon & \beta \end{array}\right]$$

is real for $\beta \ge 2\varepsilon$.

Remark 2.9 When $C = O$, a purely real spectrum may also be obtained by the classical augmented Lagrangian approach [13]. In this approach the original system is replaced by an equivalent one with coefficient matrix

$$\begin{bmatrix} A + \mu B^T B & B^T \\ -B & O \end{bmatrix}, \tag{2.10}$$

where $\mu > 0$ is a parameter. Here we assume that A is positive semidefinite, and B has full row rank with $\ker(A) \cap \ker(B) = \{0\}$. Note that $A + \mu B^T B$ is symmetric positive definite for all $\mu > 0$. The condition for the reality of the eigenvalues can always be satisfied by taking μ sufficiently large. Indeed, the largest eigenvalue of $B(A + \mu B^T B)^{-1} B^T$ tends to zero for $\mu \to \infty$, while

$$\lim_{\mu\to\infty} \lambda_{\min}(A + \mu B^T B) = \min_{u\in\ker(B)\setminus\{0\}} \frac{u^T A u}{u^T u} > 0\,;$$

see [13, Prop. 2.3]. The choice of an "optimal" value of μ requires a delicate balancing act between the conditioning properties of the (1,1) block in (2.10) and the rate of convergence of iterative solvers applied to the whole system (2.10). We refer the reader to [13] for details; see also [18] for a recent study.

Remark 2.10 In [8], the authors show how to derive *preconditioned* saddle point matrices with a positive real spectrum and identify a nonstandard inner product relative to which these preconditioned matrices are self-adjoint and positive definite. Our results show that under certain conditions, no preconditioning is necessary to obtain a positive real spectrum. Of course, preconditioning is usually necessary in practice in order to achieve rapid convergence of Krylov subspace methods.

It may happen that the symmetric matrix A has multiple eigenvalues. Next, we show that this property is partially inherited by the matrix M_-.

Proposition 2.11 *Assume that B has full rank and that M_- is diagonalizable. Let μ be an eigenvalue of A with multiplicity $k > m$. If* range(B) *does not include an eigenvector of A not associated with μ, then M_- has an eigenvalue $\lambda = \mu$ of multiplicity at most $k - m$ with a complete set of orthogonal eigenvectors. If* range(B) *includes an eigenvector of A not associated with μ, then the multiplicity of $\lambda = \mu$ may be higher.*

Proof Let $A = [Q_\mu, Q_2]\mathrm{diag}(\mu I_k, \Psi)[Q_\mu, Q_2]^T$ be the eigenvalue decomposition of A, with $[Q_\mu, Q_2]$ orthogonal and $\Psi - \mu I$ nonsingular. Let $x = [u; v]$ be an eigenvector of M_- satisfying $Bu = 0, v = 0$ and let Z be a matrix with orthonormal columns spanning the null space of B, so that $u = Zd$ for some vector d. Then from (2.2) we obtain $Au = \lambda u$ with $u = Zd$. We have

$$\begin{aligned}(\mu Q_\mu Q_\mu^T + Q_2 \Psi Q_2^T)u &= \lambda u \\ (\mu(I - Q_2 Q_2^T) + Q_2 \Psi Q_2^T) Zd &= \lambda Zd \\ Q_2(\Psi - \mu I) Q_2^T Zd &= (\lambda - \mu) Zd \\ Z^T Q_2(\Psi - \mu I) Q_2^T Zd &= (\lambda - \mu) d.\end{aligned}$$

The matrix $Z^T Q_2(\Psi - \mu I) Q_2^T Z$ is singular, and its null space has dimension equal to $\dim(\ker(Q_2^T Z)) \geq n - m - (n - k) = k - m$. Since the matrix is symmetric, the corresponding vectors d can be taken to be orthogonal, so that $u = Zd$ will also be orthogonal. If range(B) does not include an eigenvector of A not associated with μ, then $Q_2^T Z$ has full (row) rank, so that $\dim(\ker(Q_2^T Z)) = k - m$. □

The proof above shows that multiple eigenvalues equal to μ may arise if there are eigenvectors of M_- of the form $[u; 0]$, with $Bu = 0$; *cf.* Proposition 2.2.

The result of Proposition 2.11 is a significant generalization of Lemma 2.2 (ii) in [12].

2.2 Bounds for the eigenvalues of M_-

In this section we derive some simple bounds for the eigenvalues of M_-.

Proposition 2.12 *Assume $A = A^T$ is positive definite, $C = C^T$ is positive semidefinite, and B^T has full rank. Let $(\lambda, [u; v])$ be an eigenpair of (2.4) with $\|[u; v]\| = 1$. Then*

1. *If* $\Im(\lambda) \neq 0$ *then* $\|u\|^2 = \frac{1}{2} = \|v\|^2$ *and*

$$\Re(\lambda) = \frac{1}{2}\left(\frac{u^*Au}{u^*u} + \frac{v^*Cv}{v^*v}\right), \tag{2.11}$$

so that

$$\frac{1}{2}(\lambda_{\min}(A) + \lambda_{\min}(C)) \leq \Re(\lambda) \leq \frac{1}{2}(\lambda_{\max}(A) + \lambda_{\max}(C)),$$
$$|\Im(\lambda)| \leq \sigma_{\max}(B).$$

2. *If* $\Im(\lambda) = 0$ *then*

$$\lambda = \frac{u^*Au + v^*Cv}{u^*u + v^*v}, \tag{2.12}$$

so that either

$$2\min\{\lambda_{\min}(A), \lambda_{\min}(C)\} \leq \lambda \leq \max\{\lambda_{\max}(A), \lambda_{\max}(C)\}, \tag{2.13}$$

(case $v \neq 0$*), or*

$$\lambda_{\min}(A) \leq \lambda \leq \lambda_{\max}(A)$$

(case $v = 0$*).*

The same result holds if B^T *is rank deficient with* C *positive definite on* $\ker(B^T)$.

Proof A straightforward application of Bendixsons' Theorem [20, p. 69], shows that for any eigenvalue λ of M_-, the following bounds on the real and imaginary part of λ hold:

$$\min\{\lambda_{\min}(A), \lambda_{\min}(C)\} \leq \Re(\lambda) \leq \max\{\lambda_{\max}(A), \lambda_{\max}(C)\}, \quad |\Im(\lambda)| \leq \sigma_{\max}(B).$$

Note that these bounds hold for any $A = A^T$, $C = C^T$, and B. Consider now the equations in (2.2) and (2.3). We note that it cannot be that $u = 0$, otherwise, since B^T has full rank, the first equation would give $v = 0$, which cannot be true. (If B^T is rank deficient, then $u = 0$ implies that λ is a nonzero eigenvalue of C.) We can thus assume $u \neq 0$. For $v = 0$, Proposition 2.2 applies, therefore $\lambda \in \mathbb{R}$.

Let $\lambda = \lambda_1 + i\lambda_2$, with $\lambda_1, \lambda_2 \in \mathbb{R}$. Multiplying (2.4) from the left by (u^*, v^*), we obtain

$$u^*Au + v^*Cv - \lambda = -u^*B^Tv + v^*Bu.$$

The real part yields $\lambda_1 = u^*Au + v^*Cv$, which proves the result for the real part of the eigenvalue, since $\|u\|^2 + \|v\|^2 = 1$. In particular, for $\lambda_2 \neq 0$, it also holds $\|u\|^2 = \|v\|^2 = \frac{1}{2}$. □

Remark 2.13 For $C = O$, complex eigenvalues can be further bounded (*cf.* [28]). The property $\|u\| = \|v\|$ together with $-Bu = \lambda v$, yields

$$|\lambda| = \frac{\|Bu\|}{\|v\|} \leq \|B\| = \sigma_{\max}(B).$$

We note that when the eigenvalues are all real and positive, it becomes possible to use the Chebyschev algorithm instead of a symmetric Krylov solver applied to M_+ or a nonsymmetric Krylov solver applied to M_-. This may be advantageous on parallel architectures, since the Chebyschev algorithm does not require inner products, a communication-intensive operation. In order to use the Chebyschev algorithm effectively, estimates of the smallest and largest eigenvalues are needed: $a \approx \lambda_{\min}(M_-)$, $b \approx \lambda_{\max}(M_-)$ (with $a > 0$). Upper bounds are not difficult to obtain, but lower bounds may be more problematic. Proposition 2.12 may be useful to derive bounds; note, however, that when $C = O$ the lower bound from part 2 of Proposition 2.12 is generally 0. A similar difficulty arises in eigenvalues estimates for M_+, see [25].

3 Working with a definite inner product

In this section we discuss conditions for the existence of an inner product in $\mathbb{R}^{n+m}$ with respect to which M_- with $C = O$ is symmetric positive definite and therefore, in particular, diagonalizable. We also discuss the practical advantages of introducing a proper inner product for analyzing the convergence of Krylov subspace linear system solvers with M_-.

Consider the symmetric matrix

$$G = \begin{bmatrix} A - \gamma I & B^T \\ B & \gamma I \end{bmatrix}.$$

It is immediate to verify that $GM_- = M_-^T G$ for any γ. The following result shows that under certain conditions G is positive definite, and therefore it defines the sought after inner product.

Proposition 3.1 *Let A be symmetric and positive definite. Let $\gamma = \frac{1}{2}\lambda_{\min}(A)$. If $\lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$, then G is positive definite and M_- is diagonalizable.*

Proof We prove the result by showing that under the given hypotheses, all eigenvalues of G are positive. Writing the eigenvalue problem $Gz = \theta z$ blockwise and using $z^T = [x^T, y^T]$, we obtain

$$(A - \gamma I)x + B^T y = \theta x, \qquad Bx + \gamma y = \theta y.$$

If $\theta = \gamma$ then $\theta > 0$. If $\theta \neq \gamma$ from the second equation we obtain $y = (\theta - \gamma)^{-1} Bx$. Note that it must be $x \neq 0$ for otherwise $\theta = \gamma$. Substituting into the first block equation and multiplying by x^T we obtain the following quadratic equation in θ:

$$\theta^2 x^T x - \theta x^T Ax + \gamma x^T Ax - \gamma^2 x^T x - x^T B^T Bx = 0, \tag{3.1}$$

whose (real) roots are

$$\theta_\pm = \frac{1}{2}\frac{x^T Ax}{x^T x} \pm \frac{1}{2}\sqrt{\left(\frac{x^T Ax}{x^T x} - 2\gamma\right)^2 + 4\frac{x^T B^T Bx}{x^T x}}. \tag{3.2}$$

Clearly, $\theta_+ > 0$, while

$$\begin{aligned}\theta_- > 0 &\Leftrightarrow \frac{1}{2}\frac{x^T Ax}{x^T x} - \frac{1}{2}\sqrt{\left(\frac{x^T Ax}{x^T x} - 2\gamma\right)^2 + 4\frac{x^T B^T Bx}{x^T x}} > 0 \\ &\Leftrightarrow \left(\frac{x^T Ax}{x^T x}\right)^2 > \left(\frac{x^T Ax}{x^T x} - 2\gamma\right)^2 + 4\frac{x^T B^T Bx}{x^T x} \\ &\Leftrightarrow \gamma\frac{x^T Ax}{x^T x} > \gamma^2 + \frac{x^T B^T Bx}{x^T x}. \end{aligned} \tag{3.3}$$

Dividing by $\frac{x^T Ax}{x^T x}$ and setting $\left(\frac{x^T Ax}{x^T x}\right)^{-1}\frac{x^T B^T Bx}{x^T x} = \frac{x^T B^T Bx}{x^T Ax} = \frac{q^T A^{-\frac{1}{2}} B^T BA^{-\frac{1}{2}} q}{q^T q}$, with $q = A^{\frac{1}{2}}x$, the condition (3.3) above can be rewritten as

$$\gamma > \gamma^2\frac{x^T x}{x^T Ax} + \frac{q^T A^{-\frac{1}{2}} B^T BA^{-\frac{1}{2}} q}{q^T q}. \tag{3.4}$$

We now set $\gamma = \frac{1}{2}\lambda_{\min}(A)$. If $\lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$, we have

$$\begin{aligned}\gamma^2\frac{x^T x}{x^T Ax} + \frac{q^T A^{-\frac{1}{2}} B^T BA^{-\frac{1}{2}} q}{q^T q} &\le \frac{1}{4}\lambda_{\min}(A)^2\frac{1}{\lambda_{\min}(A)} + \lambda_{\max}(BA^{-1}B^T) \\ &< \frac{1}{4}\lambda_{\min}(A) + \frac{1}{4}\lambda_{\min}(A) = \frac{1}{2}\lambda_{\min}(A) = \gamma.\end{aligned}$$

The condition (3.4) is satisfied, so that $\theta_- > 0$ and all the eigenvalues of G are positive. It follows from $GM_- = M_-^T G$ with G symmetric positive definite that

$$G^{\frac{1}{2}} M_- G^{-\frac{1}{2}} = G^{-\frac{1}{2}}(M_-^T G)\, G^{-\frac{1}{2}}.$$

Therefore M_- is similar to a symmetric matrix and thus is diagonalizable. □

When $A = \eta I$, the matrix G given above reduces to the one given in [12]. Note that the condition $\lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$ cannot be relaxed, in general. For example, the matrix

$$M_- = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

satisfies $\lambda_{\min}(A) = 4\lambda_{\max}(BA^{-1}B^T)$ and therefore has real positive eigenvalues (both equal to 1). However, M_- is not diagonalizable. We also stress that the condition of Proposition 3.1 is sufficient but not necessary; as an example, the matrix in (2.7) has real and positive eigenvalues, is diagonalizable, but it does not satisfy the condition $\lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$.

Again, a diagonalizable matrix can always be obtained by using the augmented Lagrangian approach with $\mu > 0$ sufficiently large. Incidentally, we note that the condition number of the matrix of eigenvectors of (2.10) tends to 1 as $\mu \to \infty$. Indeed, the eigenvectors of (2.10) tend to those of a symmetric matrix as $\mu \to \infty$, and thus become orthogonal in the limit.

Under the hypotheses of the previous proposition, we can estimate the condition number of the positive definite matrix G in terms of the positive definiteness condition.

Corollary 3.2 *Let A be symmetric and positive definite, $\gamma = \frac{1}{2}\lambda_{\min}(A)$ and $\lambda_{\min}(A) > 4\lambda_{\max}(BA^{-1}B^T)$, so that $\rho := \frac{1}{2}\gamma - \lambda_{\max}(BA^{-1}B^T) > 0$. Then,*

$$\kappa(G) := \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} \le \frac{1}{2}\frac{\lambda_{\max}(A)}{\rho}\left(1 + \sqrt{(1-\kappa(A)^{-1})^2 + 1}\right) \approx \frac{\lambda_{\max}(A)}{\rho} \tag{3.5}$$

Proof Let $\theta > 0$ be an eigenvalue of G and set $\lambda_{\min} = \lambda_{\min}(A)$, $\lambda_{\max} = \lambda_{\max}(A)$ for short. From (3.2) we obtain

$$\begin{aligned}\theta &\le \frac{1}{2}\frac{x^TAx}{x^Tx}\left(1 + \sqrt{\left(1 - \lambda_{\min}\frac{x^Tx}{x^TAx}\right)^2 + 4\frac{x^TB^TBx}{x^TAx}\frac{x^Tx}{x^TAx}}\right)\\ &\le \frac{1}{2}\lambda_{\max}\left(1 + \sqrt{(1-\kappa(A)^{-1})^2 + 1}\right).\end{aligned}$$

For the lower bound, using (3.1) and dividing by x^TAx we obtain

$$\theta^2\frac{x^Tx}{x^TAx} + \gamma = \theta + \gamma^2\frac{x^Tx}{x^TAx} + \frac{x^TB^TBx}{x^TAx}$$

$$\theta^2\frac{x^Tx}{x^TAx} + \frac{1}{2}\gamma + \left(\frac{1}{2}\gamma - \frac{x^TB^TBx}{x^TAx}\right) = \theta + \gamma^2\frac{x^Tx}{x^TAx}. \tag{3.6}$$

The right-hand side in (3.6) can be bounded from above as $\theta + \gamma^2\frac{x^Tx}{x^TAx} \le \theta + \gamma^2/\lambda_{\min} = \theta + 1/4\lambda_{\min}$, while using $\rho > 0$ the left-hand side of (3.6) can be estimated from below as $\theta^2\frac{x^Tx}{x^TAx} + \frac{1}{2}\gamma + \left(\frac{1}{2}\gamma - \frac{x^TB^TBx}{x^TAx}\right) \ge \theta^2\frac{x^Tx}{x^TAx} + \frac{1}{2}\gamma + \rho \ge \frac{1}{2}\gamma + \rho$. Collecting the two estimates for the equality above and recalling that $1/4\lambda_{\min} = 1/2\gamma$ yields

$$\frac{1}{2}\gamma + \rho \le \theta + \frac{1}{4}\lambda_{\min}, \quad \text{or } \rho \le \theta.$$

This result provides the lower bound for θ, from which the final result follows. The approximation in (3.5) holds for $\kappa(A)$ large. □

Let us now assume that the conditions of Proposition 3.1 are satisfied so that G is SPD and M_- is diagonalizable with real, positive eigenvalues. Let Λ, X be the eigenvalue and eigenvector matrices of M_-, respectively. Then we have

$$M_-X = X\Lambda \quad \Leftrightarrow \quad GM_-X = GX\Lambda \quad \Leftrightarrow \quad M_-^TGX = GX\Lambda.$$

Since G is SPD and M_-^TG is symmetric, with Λ diagonal and positive, the columns of X are G-orthogonal, that is it holds $X^TGX = D$ with D diagonal and positive definite. In particular, we have $X^{-T} = GXD^{-1}$.

We want to characterize the role of G in the convergence of Krylov subspace solvers applied to linear systems of type $M_-z = b$. We start by observing that $G^{\frac{1}{2}}XD^{-1}X^TG^{\frac{1}{2}} = I_n$, so that, for any vector $w \in \mathbb{R}^{n+m}$ we have

$$\|D^{\frac{1}{2}}X^{-1}w\| = \|D^{-\frac{1}{2}}X^TGw\| = \|(D^{-\frac{1}{2}}X^TG^{\frac{1}{2}})G^{\frac{1}{2}}w\| = \|G^{\frac{1}{2}}w\| = \|w\|_G. \tag{3.7}$$

Let r_m be the residual obtained by a Krylov subspace method in $K_m(M_-, r_0)$, with $r_0 = b - M_- z_0$ and z_0 an initial approximation. Then $r_m = p(M_-)r_0$ for some polynomial p of degree not greater than m. Since M_- is diagonalizable, we have $r_m = Xp(\Lambda)X^{-1}r_0$, or, equivalently,

$$X^{-1}r_m = p(\Lambda)X^{-1}r_0 \quad \Leftrightarrow \quad D^{\frac{1}{2}}X^{-1}r_m = p(\Lambda)D^{\frac{1}{2}}X^{-1}r_0,$$

with $\|D^{\frac{1}{2}}X^{-1}r_m\| \leq \|p(\Lambda)\| \, \|D^{\frac{1}{2}}X^{-1}r_0\|$. Using (3.7) we obtain

$$\|r_m\|_G \leq \|p(\Lambda)\| \, \|r_0\|_G \quad \text{from which} \quad \|r_m\| \leq \kappa(G)^{\frac{1}{2}}\|p(\Lambda)\| \, \|r_0\|. \quad (3.8)$$

Therefore, the convergence rate of the considered method can be completely analyzed either in terms of the G-inner product, or in terms of the conditioning of an appropriate SPD matrix G, together with the behavior of the polynomial p on the set of the (real) eigenvalues of M_-; we refer to [1] for more details on the relation between G-inner product and Euclidean inner product.

The second bound in (3.8) should be compared with the classical bound $\|r_m\| \leq \|X\| \, \|X^{-1}\| \|p(\Lambda)\| \, \|r_0\|$ for general diagonalizable M_- (see, e.g., [26]), where the conditioning of the eigenvector matrix and its estimate play a crucial role in the sharpness of the convergence bound. Thanks to (3.7) we have been able to replace $\|X\| \, \|X^{-1}\|$ with a more insightful quantity, the conditioning of G. If a good estimate of $\lambda_{\min}(A)$ is available, then the Conjugate Gradient method may be employed for solving systems with M_- by using the inner product defined by G (see, e.g., [2]). Note that multiplications with G do not entail significant extra computational cost, since G can be written as $G = M_+ - \gamma J$ (or $G = J(M_- - \gamma I)$), and products with M_+ or M_- are already available during the iteration.

In the absence of a definite inner product, Theorem 2.1 still ensures that the eigenvector matrix is J-orthogonal, that is

$$X^* J X = S, \quad (3.9)$$

where S is block diagonal with 2×2 blocks. More precisely, the 2×2 blocks that are not diagonal are associated with complex conjugate eigenpairs (λ, x) and $(\bar{\lambda}, \bar{x})$, for which it holds $x^* J x = 0$ and $\bar{x}^* J x \neq 0$. For real eigenvalues, the corresponding portion of S is diagonal with nonzero diagonal elements. This is the case also for multiple real eigenvalues, since we assume that M_- is diagonalizable [17, p. 36].

Setting $x = [u; v]$, $\|x\| = 1$, for real eigenvalues we define the quantity

$$s_{\rm r} := x^* J x = \|u\|^2 - \|v\|^2 \in \mathbb{R}.$$

If λ is a non-real eigenvalue, then $\|u\| = \|v\|$, and we define

$$s_{\rm c} := \bar{x}^* J x = \bar{u}^* u - \bar{v}^* v \in \mathbb{C}.$$

Therefore, the matrix S has the following structure,

$$S = \begin{bmatrix} S_1(\lambda_1) & & & & \\ & S_2(\lambda_2) & & & \\ & & \ddots & & \\ & & & s_{\rm r}(\lambda_{n+m-1}) & \\ & & & & s_{\rm r}(\lambda_{n+m}) \end{bmatrix}, \quad S_i(\lambda_i) = \begin{bmatrix} 0 & \bar{s}_{\rm c}(\lambda_i) \\ s_{\rm c}(\lambda_i) & 0 \end{bmatrix},$$

where at most m diagonal blocks $S_1, \dots, S_m$ are associated with complex eigenvalues (*cf.* Proposition 2.3). Using (3.9), we have $X^{-1} = S^{-1}X^*J$, therefore the condition number of X can be written as

$$\|X\| \, \|X^{-1}\| = \|X\| \, \|S^{-1}X^*J\| \le \|X\|^2 \|S^{-1}\|.$$

Clearly, a possible ill-conditioning of X depends on $\|S^{-1}\|$, but may also depend on the dimension of the problem with $\|X\|$. Our (somewhat limited) numerical experience seemed to indicate that the problem dimension did not play a significant role. Nonetheless, a complete spectral analysis in the presence of nonreal eigenvalues remains to be done. Finally, a short-term recurrence method such as simplified QMR may be exploited by using either the J-symmetry or the G-symmetry of matrix M_- [14].

4 An example: the Stokes problem

In this section we use saddle point systems arising from the classical Stokes problem to illustrate our theory. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be bounded and have a sufficiently smooth boundary Γ. A fundamental problem in fluid mechanics is the so-called *generalized Stokes problem* [15,32]:

$$\alpha \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{4.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{4.2}$$

$$\mathcal{B}\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma. \tag{4.3}$$

Here $\mathbf{f}$ is a given external force field (like gravity), $\mathbf{u}$ denotes the velocity vector field, p is the pressure scalar field, and $\mathcal{B}$ represents some type of boundary operator (e.g., a trace operator for Dirichlet boundary conditions). The parameter α is zero for the steady-state Stokes problem, and proportional to $1/\Delta t$ (where Δt is the time step) in the unsteady case when the time derivative $\mathbf{u}_t$ is treated by implicit schemes, such as backward Euler. The constant $\nu > 0$ represents viscosity; if $\alpha = 0$, we can always rescale the problem and assume that $\nu = 1$.

Div-stable discretizations of the Stokes system (such as MAC, or mixed finite element methods satisfying the LBB condition [10]) lead to linear systems of equations of the form

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \qquad \text{or } M_+ x = b. \tag{4.4}$$

Here A is positive definite and is a direct sum of d matrices of the form $F = \alpha\, M_u + \nu\, L$, where M_u is a (velocity) mass matrix (or a scaled identity for finite difference schemes like MAC) and L represents a discretization of the (negative) Laplacian. Moreover, B is a discrete divergence operator and B^T a discrete gradient. Unless some additional condition is imposed on the pressure (*e.g.*, $\int_\Omega p\, dx = 0$), B^T is rank deficient with a one-dimensional null space $\ker(B^T) = \text{span}\{e\}$, where e is the vector of all ones. For unstable discretizations a stabilization term C is added, leading to a coefficient matrix of the form M_+, with C positive semidefinite, $C \ne O$. Let us now consider the associated nonsymmetric matrix M_-. Numerical experiments reveal that for a number of spatial discretizations of the generalized

Stokes problem (under a variety of boundary conditions), the eigenvalues of M_- are all real (and, of course, positive), provided that the viscosity ν is not too small. In particular, the eigenvalues are all real when $\alpha = 0$ and $\nu = 1$. Furthermore, M_- is diagonalizable. Indeed, the matrix G described in section 3 is SPD and therefore a conjugate gradient-type method (i.e., a Krylov subspace method based on short recurrences) exists for this problem.[1] To our knowledge, such properties of the spectrum of M_- have not been explicitly observed or proved before. Some analysis has been given in [28], but neither the reality of the eigenvalues nor the diagonalizability property for the discrete Stokes problem were noted there.

The fact that M_- is diagonalizable and has all the eigenvalues positive and real can be explained by applying the general results in sections 2.1 and 3. Consider for instance the case where $\Omega = [0, 1] \times [0, 1]$. Assuming zero Dirichlet boundary conditions, the smallest eigenvalue of the (negative) Laplacian is given by $\lambda_{\min} = 2\pi^2 \approx 19.74$. Therefore, for a sufficiently fine spatial discretization, the smallest eigenvalue of the (1,1) block A in M_- satisfies

$$\lambda_{\min}(A) \approx \alpha + 2\nu\pi^2 \approx \alpha + (19.74)\nu.$$

For the stationary Stokes problem we can take $\alpha = 0$, $\nu = 1$; for a div-stable discretization (like MAC) we have $\|S\| \approx 1$ and the reality condition (with $C = O$) is clearly satisfied, since $19.74 > 4$. For unstationary problems the reality condition will be satisfied subject to certain restrictions on α, ν. It is easy to see that for a fixed α, it is $\|S\| \approx \nu^{-1}$ and the reality condition, for h sufficiently small, is $\alpha + 2\nu\pi^2 \geq 4\nu^{-1}$. (In practice, we found that the eigenvalues are all real even for the coarsest meshes, e.g., for $h = 0.25$.) The condition is certainly satisfied if $\nu \geq \sqrt{2}/\pi \approx 0.45$. Also, for a fixed ν the eigenvalues will be all real provided that α is large enough, i.e., for sufficiently small time steps. Similar conclusions apply to the case $C = \beta I_m$, $\beta > 0$ and in fact also for more general forms of the stabilization term. Under the hypotheses of Proposition 3.1, the convergence of Krylov subspace methods applied to the nonsymmetric formulation may be fully described in terms of the spectrum of M_- and of G; *cf.* (3.8) and the discussion around it.

We also performed a few numerical experiments aimed at verifying the bound on $\kappa(G)$ given in Corollary 3.2, and we found that the upper bound and its approximation are indeed pretty good estimates of the spectral condition number of G. The results are reported in the table below.

Grid size	$\kappa(G)$	Bound in (3.5)	Estimate in (3.5)	GMRES(M_-) # its	MINRES(M_+) # its
8×8	91	193	161	65	65
16×16	337	703	584	148	149
32×32	1291	2667	2211	352	391
64×64	5040	10364	8587	783	852

For the sake of completeness, in the table we also report the number of GMRES and MINRES iterations to solve with M_- and M_+, respectively, for a final residual tolerance of 10^{-6}. In practice, these iterations would be used with preconditioning.

[1] We emphasize that this possibility is more theoretical than practical, since it is not clear at present how to derive preconditioners that are self-adjoint and positive definite with respect to the G-inner product.

Since we are only interested here in checking the goodness of our estimates, we do not consider preconditioning. The asymptotic rate of convergence of MINRES when applied to the considered Stokes problem with M_+ is known to be $1 - ch^{3/2}$ (see [35, formula (5.3)]), where h is the mesh size and c is a modest constant. This is confirmed by our experiments, that show a superlinear increase in the number of MINRES iterations. For the case of M_-, the number of iterations becomes lower than for MINRES when refining the grid. Using (3.8), we can see that the GMRES minimum residual polynomial p satisfies

$$\|p(\Lambda)\| \leq 2\left(\frac{\sqrt{\kappa_M}-1}{\sqrt{\kappa_M}+1}\right)^m,$$

where $\kappa_M = \lambda_{\max}(M_-)/\lambda_{\min}(M_-) > 1$. Using (2.13), $\kappa_M \approx \kappa(A) = a^2h^{-2}$, for some positive constant a, so that

$$\left(\frac{\sqrt{\kappa_M}-1}{\sqrt{\kappa_M}+1}\right) \approx \left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right) = \frac{a-h}{a+h} = 1 - \frac{2h}{1+h}.$$

This last quantity represents the asymptotic rate of convergence for the 2-norm of the residual in (3.8), since $\kappa(G)^{1/m} \to 1$ as $m \to \infty$. Clearly, we do not advocate using GMRES with M_- for practical purposes, as sub-optimal but cheaper methods should be preferred; see the discussion in section 3. Nonetheless, the results above show that for fine grids, working with M_- may provide some advantages, since $1-2h/(1+h)$ is smaller than $1-h^{3/2}$ for $0 < h < 1$. A more detailed polynomial approximation analysis of M_+ and M_- for $A = \eta I_n$ can be found in [11].

5 Applications to preconditioning

As already mentioned, the linear system (4.4) can be solved using Krylov subspace methods for symmetric indefinite problems, such as MINRES or SYMMLQ [23]. Preconditioning, however, is mandatory for fast convergence. In the last several years, a number of preconditioning techniques have been developed for solving saddle point problems; see [7] for a survey. Among the most popular techniques we mention block diagonal, block triangular, and indefinite (constraint) preconditioners. We further mention preconditioners specifically developed for the nonsymmetric formulation

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix}; \tag{5.1}$$

see [5,6,28]. In the remainder of this section we use our general framework to analyze spectral properties of the preconditioned matrices corresponding to various types of preconditioners.

5.1 Block diagonal preconditioning

Both MINRES and SYMMLQ can be used with preconditioning, provided that the preconditioner is symmetric positive definite. The preconditioned matrix is then congruent to the unpreconditioned one, and therefore it has the same number of positive and negative eigenvalues as the latter matrix (Sylvester's Law of Inertia). Descriptions of SPD preconditioners for saddle point problems can be found, for example, in [7,10,22,24,25]. These preconditioners are block diagonal matrices of the form

$$P_+ = \begin{bmatrix} \widehat{A} & O \\ O & \widehat{S} \end{bmatrix},$$

where $\widehat{A}$ and $\widehat{S}$ are symmetric positive definite approximations to A and $S = C + BA^{-1}B^T$, respectively. In the case of the generalized Stokes problem, spectrally equivalent approximations $\widehat{A}$ and $\widehat{S}$ are known that lead to *asymptotically optimal* preconditioners, i.e., preconditioners that result in rates of convergence independent of the discretization parameter h; see, *e.g.*, [9,10,21,29].

One can also consider indefinite block diagonal preconditioners of the form

$$P_- = \begin{bmatrix} \widehat{A} & O \\ O & -\widehat{S} \end{bmatrix},$$

in which case the preconditioned matrix may have nonreal eigenvalues. Preconditioners of the form $P_\pm$ with $\widehat{A} = \frac{1}{\eta^2}A$ ($\eta > 0$) and $\widehat{S} \approx S$ (a symmetric and positive definite approximation to S) have been studied in [12] for the case $C = O$. If L denotes the Cholesky factor of A and L_s the Cholesky factor of $\widehat{S}$, then symmetrically applying the preconditioner $P_\pm$ and dividing through by η results in preconditioned matrices of the form

$$\begin{bmatrix} \eta I_n & \widehat{B}^T \\ \pm\widehat{B} & O \end{bmatrix},$$

where $\widehat{B} = L_s^{-1}BL^{-T}$. The nonsymmetric matrix $\widehat{M}_-$ has all its eigenvalues real and positive provided that

$$\eta^2 \geq 4\lambda_{\max}(L_s^{-1}SL_s^{-T}).$$

We consider two extreme cases. If L_s is the exact Cholesky factor of $S = BA^{-1}B^T$, a sufficient condition for the eigenvalues to be all real and positive becomes simply $\eta \geq 2$. If, on the other hand, $L_s = I_m$ (that is, $\widehat{S} = I_m$), then we have that the condition for real eigenvalues is $\eta^2 \geq 4\lambda_{\max}(S)$. For div-stable discretizations of the Stokes problem (see section 4) we can assume $\lambda_{\max}(S) = 1$ and therefore a sufficient condition for a real positive spectrum is, again, $\eta \geq 2$. Also, $\eta > 2$ guarantees that the preconditioned matrix is diagonalizable. These conditions are sufficient, but not necessary; in practice, a real spectrum may occur for smaller values of η. However, our bound is quite sharp for Stokes: in [12], it was found that a real spectrum occurred for $\eta > 1.9862$.

It was shown in [12] that the choice of η has little effect on the convergence of Krylov methods preconditioned by block diagonal preconditioners of the form

$P_\pm$. In the same paper it is shown that it is more efficient to use MINRES with the positive definite preconditioner P_+ than a method like QMR or GMRES with the indefinite preconditioner P_-.

5.2 Inexact constraint preconditioning

Constraint preconditioners are another important class of preconditioners for saddle point problems; see [7, section 10.2] for a survey. In this case MINRES or SYMMLQ should be replaced by a Krylov method that can accommodate symmetric indefinite preconditioning, like simplified QMR [14]. In alternative, a nonsymmetric Krylov subspace method like GMRES [27] or Bi-CGSTAB [33] can always be used, but these algorithms do not exploit symmetry in the system or in the preconditioner.

Constraint preconditioners usually take the form (for the case $C = O$)

$$P_I = \begin{bmatrix} \widehat{A} & B^T \\ B & O \end{bmatrix},$$

where $\widehat{A}$ is an approximation of A. This type of preconditioner is particularly efficient when A represents the discretization of a zeroth order operator such as the mass matrix. In this case, simply taking $\widehat{A} = \mathrm{diag}(A)$ yields an effective approximation for the (1,1) block. Here we assume the problem has been scaled so that $\mathrm{diag}(A) = I_n$, and we let $\widehat{A} = I_n$. We write the *inexact* constraint preconditioner as

$$P_I = \begin{bmatrix} I_n & O \\ B & I_m \end{bmatrix} \begin{bmatrix} I_n & O \\ O & -H \end{bmatrix} \begin{bmatrix} I_n & B^T \\ O & I_m \end{bmatrix}, \quad H \approx BB^T,$$

and H symmetric and positive definite. (For $H = BB^T$ we obtain the exact constraint preconditioner, which is often too expensive to be practical.) Then the eigenvalue problem

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda P_I \begin{bmatrix} x \\ y \end{bmatrix}$$

can be written as

$$\begin{bmatrix} I_n & O \\ -B & I_m \end{bmatrix} \begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} I_n & -B^T \\ O & I_m \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} I_n & O \\ O & -H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

where $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I_n & B^T \\ O & I_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Explicitly computing the left-hand side matrix yields

$$\begin{bmatrix} A & (I_n - A)B^T \\ B(I_n - A) & -B(2I_n - A)B^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} I_n & O \\ O & -H \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{5.2}$$

a generalized eigenproblem of the form (2.9) if $B(2I_n - A)B^T$ is a positive semidefinite matrix. Note that this last condition can be fulfilled by scaling A so that all its eigenvalues are not larger than 2. Next, we employ the results of Proposition 2.12 to obtain bounds for the eigenvalues of the preconditioned matrix $P_I^{-1}M_+$. To the best of our knowledge, these appear to be the first quantitative spectral bounds for $P_I^{-1}M_+$, when P_I is an *inexact* constraint preconditioner, namely when $H \neq BB^T$.

Corollary 5.1 *Let λ be an eigenvalue of the pencil (M_+, P_I) and let $H = R^T R$ be the Cholesky factorization of H. Assume that the matrix $B(2I_n - A)B^T$ is positive semidefinite, and let $S = R^{-T} B(2I_n - A)B^T R^{-1}$. Then*

1. *If $\Im(\lambda) \neq 0$, then*

$$|\Im(\lambda)| \leq \|(I_n - A)BR^{-1}\|,$$
$$\frac{1}{2}(\lambda_{\min}(A) + \lambda_{\min}(S)) \leq \Re(\lambda) \leq \frac{1}{2}(\lambda_{\max}(A) + \lambda_{\max}(S))$$

2. *If $\Im(\lambda) = 0$ then either*

$$2\min\{\lambda_{\min}(A), \lambda_{\min}(S)\} \leq \lambda \leq \max\{\lambda_{\max}(A), \lambda_{\max}(S)\}$$

for $v \neq 0$, or $\lambda_{\min}(A) \leq \lambda \leq \lambda_{\max}(A)$ for $v = 0$.

Proof The proof immediately follows from Proposition 2.12 applied to the pencil in (5.2). □

We show the qualitative behavior of these bounds for a 2088×2088 linear system stemming from mixed finite element discretization of the 2D electrostatic problem, for which the indefinite preconditioner has been shown to be particularly effective [24]. We refer to [24] for a detailed description of the test problem. Here we approximate BB^T of size 816 with $R^T R$, where R is the upper triangular factor of the incomplete Cholesky factorization of BB^T computed using the Matlab function `cholinc` with dropping threshold 10^{-2}. The exact eigenvalues of $P_I^{-1}M_+$ and the estimates provided by Corollary 5.1 are displayed in Figure 1. The large box shows the location of the nonreal eigenvalues, whereas the left and right vertical bars next to the real axis, denoted by a_1 and a_2 respectively, show the bounds for the real eigenvalues. The bounds for the imaginary part is only twice the largest imaginary part among the complex eigenvalues. More precisely, referring to the quantities in Corollary 5.1, we found $\|(I_n - A)BR^{-1}\| \approx 1.22$, while the eigenvalues of A and S are bounded as $0.002 \leq \lambda(A) \leq 2.38$ and $0.13 \leq \lambda(S) \leq 1.69$.

Simple algebraic manipulations of the eigenvalue problem in (5.2) provide more insightful information on the effect of using H to approximate BB^T. Indeed, after changing sign in the second block row, we can write the left-hand matrix in (5.2) as

$$\begin{aligned}
\begin{bmatrix} A & (I_n - A)B^T \\ -B(I_n - A) & B(2I_n - A)B^T \end{bmatrix} &= \begin{bmatrix} (A - I_n) + I_n & (I_n - A)B^T \\ -B(I_n - A) & B(I_n - A)B^T + BB^T \end{bmatrix} \\
&= \begin{bmatrix} (A - I_n) & (I_n - A)B^T \\ -B(I_n - A) & B(I_n - A)B^T \end{bmatrix} + \begin{bmatrix} I_n & O \\ O & BB^T \end{bmatrix} \\
&= \begin{bmatrix} I_n \\ B \end{bmatrix} (A - I_n) \begin{bmatrix} I_n, -B^T \end{bmatrix} + \begin{bmatrix} I_n & O \\ O & BB^T \end{bmatrix}.
\end{aligned}$$

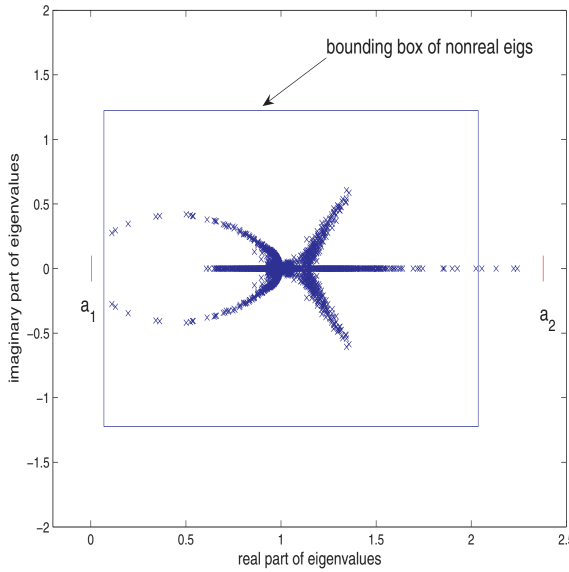

Fig. 1 Constraint preconditioning. Exact eigenvalues and bounds as in Corollary 5.1 for the electrostatic problem. The bounds for real eigenvalues are $a_1 = 0.0042$, $a_2 = 2.37$

Note that in the last expression, the first matrix has rank at most n. By setting $BB^T = H + E$, the eigenvalue problem (5.2) can be transformed into

$$\left(\begin{bmatrix} I_n \\ B \end{bmatrix}(A - I_n)\begin{bmatrix} I_n, -B^T \end{bmatrix} + \begin{bmatrix} O & O \\ O & E \end{bmatrix}\right)\begin{bmatrix} u \\ v \end{bmatrix} = (\lambda - 1)\begin{bmatrix} I_n & O \\ O & H \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

For $H = BB^T$, that is $E = O$, we obtain the known result that $2m$ eigenvalues are equal to 1, while the remaining (all real) eigenvalues satisfy $Au = \lambda u$ for $u \neq 0$ with $Bu = 0$; see [7,24] and references therein. For $E \neq O$, all eigenvector blocks u with $Bu \neq 0$ may give rise to nonreal eigenvalues, whose imaginary part can be bounded as in Corollary 5.1. In the eigenproblem above, we can also notice that a nonzero matrix E dramatically affects the null space of the low rank matrix, which corresponds to the eigenvalue $\lambda = 1$ in the exactly preconditioned problem. This fact can be appreciated in Figure 1, where most complex eigenvalues are perturbations of the unit eigenvalue, induced by the nonzero matrix E.

5.3 HSS preconditioning

Other preconditioners have been proposed specifically for the nonsymmetric formulation (5.1); see [5,6,28]. For example, the Hermitian / Skew-Hermitian splitting (HSS) preconditioner [4,6,30] is defined as follows. Let $I = I_{n+m}$ and let $\rho > 0$ be a real parameter. Define

$$P_\rho = \frac{1}{2\rho}(H + \rho I)(K + \rho I),$$

where

$$H = \frac{1}{2}(M_- + M_-^T) = \begin{bmatrix} A & O \\ O & C \end{bmatrix}, \quad K = \frac{1}{2}(M_- - M_-^T) = \begin{bmatrix} O & B^T \\ -B & O \end{bmatrix}.$$

It was shown in [5] and, under more general assumptions, in [30] that all the eigenvalues of the preconditioned matrix $P_\rho^{-1}M_-$ are real for ρ sufficiently small. In [30] we observed (experimentally) that for the discretized Stokes problem, the eigenvalues of $P_\rho^{-1}M_-$ are actually real for *all* values of $\rho > 0$. We are now in a position to rigorously prove this fact.

Let $(\lambda, [u; v])$ be an eigenpair of the preconditioned problem for $C = O$. It was shown in [30], formula (2.4), that if $\theta = \lambda\rho/(2-\lambda)$, then $(\theta, [u; v])$ is an eigenpair of the problem

$$\begin{bmatrix} A + \frac{1}{\rho}AB^TB & B^T \\ -B & O \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \theta \begin{bmatrix} u \\ v \end{bmatrix}, \tag{5.3}$$

and conversely. Hence, the eigenvalues of $P_\rho^{-1}M_-$ are real if and only if the eigenvalues of the matrix

$$T = \begin{bmatrix} A + \frac{1}{\rho}AB^TB & B^T \\ -B & O \end{bmatrix} = \begin{bmatrix} AF & B^T \\ -B & O \end{bmatrix}, \quad F = I_n + \frac{1}{\rho}B^TB$$

are real.

Note that F is SPD and commutes with B^TB (and so do F^{-1} and $F^{-\frac{1}{2}}$). Arguing as in the proof of Proposition 2.5 (with $\beta = 0$ and A replaced by AF) we obtain the equation

$$\theta^2 u - \theta AFu + B^TBu = 0$$

for the eigenvalue θ. Performing the change of variable $w = Fu$ and multiplying on the left by w^* we obtain the quadratic equation

$$\theta^2 w^*F^{-1}w - \theta w^*Aw + w^*B^TBF^{-1}w = 0,$$

which has real coefficients since F, A and B^TBF^{-1} are all real symmetric. The roots will be real provided that

$$\left(\frac{w^*Aw}{w^*F^{-1}w}\right)^2 \geq 4\,\frac{w^*B^TBF^{-1}w}{w^*F^{-1}w}. \tag{5.4}$$

Letting $z = F^{-\frac{1}{2}}w$, we can rewrite (5.4) as

$$\left(\frac{z^*F^{\frac{1}{2}}AF^{\frac{1}{2}}z}{z^*z}\right)^2 \geq 4\,\frac{z^*B^TBz}{z^*z}. \tag{5.5}$$

Condition (5.5) is equivalent to

$$\frac{z^*F^{\frac{1}{2}}AF^{\frac{1}{2}}z}{z^*z} \geq 4\,\frac{z^*B^TBz}{z^*F^{\frac{1}{2}}AF^{\frac{1}{2}}z} = 4\,\frac{q^*(A^{-\frac{1}{2}}F^{-\frac{1}{2}}B^TBF^{-\frac{1}{2}}A^{-\frac{1}{2}})q}{q^*q}, \tag{5.6}$$

where $q = A^{\frac{1}{2}} F^{\frac{1}{2}} z$. We have

$$\frac{z^* F^{\frac{1}{2}} A F^{\frac{1}{2}} z}{z^* z} \geq \lambda_{\min}(F^{\frac{1}{2}} A F^{\frac{1}{2}}) = \lambda_{\min}(AF) = \lambda_{\min}\left(A + \frac{1}{\rho} A B^T B\right)$$

and

$$\begin{aligned} \frac{q^*(A^{-\frac{1}{2}} F^{-\frac{1}{2}} B^T B F^{-\frac{1}{2}} A^{-\frac{1}{2}}) q}{q^* q} &\leq \lambda_{\max}(A^{-\frac{1}{2}} F^{-\frac{1}{2}} B^T B F^{-\frac{1}{2}} A^{-\frac{1}{2}}) \\ &= \lambda_{\max}(B(AF)^{-1} B^T) \\ &= \lambda_{\max}\left(B\left(A + \frac{1}{\rho} A B^T B\right)^{-1} B^T\right). \end{aligned}$$

Observing that (for all $\rho > 0$) the eigenvalues of $BA^{-1}B^T$ are bounded below by those of $B(A + \frac{1}{\rho} AB^T B)^{-1} B^T$ and that the eigenvalues of A are bounded above by those of $A + \frac{1}{\rho} AB^T B$, it follows that T satisfies the condition for the reality of the eigenvalues if M_- does. In particular, for the Stokes problem discussed in section 4 the preconditioned matrix has only real eigenvalues, for all $\rho > 0$. We have thus proved the following result.

Proposition 5.2 *Assume the same notation as in Proposition 2.5 holds. If* $\lambda_{\min}(A) \geq 4\,\lambda_{\max}(BA^{-1}B^T)$*, then all eigenvalues of the HSS-preconditioned matrix* $P_\rho^{-1} M_-$ *are real, for all* $\rho > 0$.

Bounds on the eigenvalues of $P_\rho^{-1} M_-$, including clustering results, can be found in [30]. Finally, we remark that the matrix T above is symmetric with respect to the indefinite symmetric matrix

$$\begin{bmatrix} F & O \\ O & -(I + \frac{1}{\rho} BB^T) \end{bmatrix},$$

so that the theory of the previous sections applies.

6 Conclusions

In this paper we have investigated the spectral properties of a class of nonsymmetric matrices M_- with a special 2×2 block structure. Such matrices arise in the numerical solution of saddle point problems. We have obtained sufficient conditions for the eigenvalues of M_- to be real and positive, and for the matrices to be diagonalizable. In many cases such conditions can be satisfied, at least in principle, by appropriate scalings or by augmented Lagrangian techniques. Positive real eigenvalues together with diagonalizability is equivalent to the existence of a non-standard inner product relative to which M_- is SPD and therefore there exists a conjugate gradient method to solve linear systems involving M_-. We have given an explicit expression for an SPD matrix G that generates such inner product. Furthermore, we have derived eigenvalue bounds for M_- and a lower bound on the spectral condition number of G which can be used to estimate the rate of convergence of the (non-standard) conjugate gradient iteration. Implications of the theory for various types of preconditioners have been discussed. We have illustrated some of our results using matrices arising from the numerical solution of the generalized Stokes problem. An interesting open question is the derivation of G-symmetric, positive definite preconditioners.

Acknowledgements We gladly acknowledge helpful discussions with Howard Elman, Eric de Sturler, Rich Lehoucq and Joerg Liesen. Thanks also to Beresford Parlett and two anonymous referees for their careful reading of the manuscript.

References

1. Ashby, S.F., Holst, M.J., Manteuffel, T.A., Saylor, P.E.: The role of the inner product in stopping criteria for Conjugate Gradient iterations, BIT, **41**, 26–52 (2001)
2. Ashby, S.F., Manteuffel, T.A., Saylor, P.E.: A taxonomy for Conjugate Gradient methods. SIAM J. Numer. Anal., **27**, 1542–1568 (1990)
3. Axelsson, O.: Iterative Solution Methods. Cambridge University Press, Cambridge, 1994
4. Bai, Z.-Z., Golub, G.H., Ng, M.K.: Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. SIAM J. Matrix Anal. Appl., **24**, 603–626 (2003)
5. Benzi, M., Gander, M.J., Golub, G.H.: Optimization of the Hermitian and skew-Hermitian splitting iteration for saddle-point problems. BIT **43**, 881–900 (2003)
6. Benzi, M., Golub, G.H.: A preconditioner for generalized saddle point problems. SIAM J. Matrix Anal. Appl. **26**, 20–41 (2004)
7. Benzi, M., Golub, G.H., Liesen, J.: Numerical solution of saddle point problems. Acta Numerica **14**, 1–137 (2005)
8. Bramble, J.H., Pasciak, J.E.: A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. Math. Comp. **50**, 1–17 (1988)
9. Bramble, J.H., Pasciak, J.E.: Iterative techniques for time-dependent Stokes problems. Comput. Math. Appl. **33**, 13–30 (1997)
10. Elman, H.C., Silvester, D.J., Wathen, A.J.: Finite Elements and Fast Iterative Solvers, Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, UK, 2005
11. Fischer, B., Peherstorfer, F.: Chebyshev approximation via polynomial mappings and the convergence behaviour of Krylov subspace methods. Electr. Trans. Numer. Anal., **12**, 205–215 (2001)
12. Fischer, B., Ramage, A., Silvester, D.J., Wathen, A.J.: Minimum residual methods for augmented systems. BIT **38**, 527–543 (1998)
13. Fortin, M., Glowinski, R.: Augmented Lagrangian Methods: Application to the Solution of Boundary-Value Problems, Stud. Math. Appl., Vol. 15, North-Holland, Amsterdam, 1983
14. Freund, R.W., Nachtigal, N.M.: Software for simplified Lanczos and QMR algorithms. Appl. Numer. Math. **19**, 319–341 (1994)
15. Glowinski, R.: Numerical Methods for Fluids (Part 3). Finite Element Methods for Incompressible Viscous Flow. In: P.G. Ciarlet and J.L. Lions (eds.,) Handbook of Numerical Analysis, Vol. IX, North-Holland, Elsevier, Amsterdam etc., 2003
16. Gohberg, I., Lancaster, P., Rodman, L.: Matrix Polynomials. Academic Press, New York, London, 1982
17. Gohberg, I., Lancaster, P., Rodman, L.: Matrices and Indefinite Scalar Products, Operator Theory: Advances and Applications, Birkhäuser, Basel, 1983
18. Golub, G.H., Greif, C.: On solving block-structured indefinite linear systems. SIAM J. Sci. Comput. **24**, 2076–2092 (2003)
19. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, 1985
20. Householder, A.S.: The Theory of Matrices in Numerical Analysis, Blaisdell Publishing Co., New York, 1964. Reprinted by Dover, New York, 1975
21. Mardal, K.-A., Winther, R.: Uniform preconditioners for the time dependent Stokes problem. Numer. Math. **98**, 305–327 (2004)

22. Murphy, M.F., Golub, G.H., Wathen, A.J.: A note on preconditioning for indefinite linear systems. SIAM J. Sci. Comput. **21**, 1969–1972 (2000)
23. Paige, C.C., Saunders, M.A.: Solution of sparse indefinite systems of linear equations. SIAM J. Numer. Anal. **12**, 617–629 (1975)
24. Perugia, I., Simoncini, V.: Block-diagonal and indefinite symmetric preconditioners for mixed finite element formulations. Numer. Linear Algebra Appl. **7**, 585–616 (2000)
25. Rusten, T., Winther, R.: A preconditioned iterative method for saddlepoint problems. SIAM J. Matrix Anal. Appl. **13**, 887–904 (1992)
26. Saad, Y.: Iterative Methods for Sparse Linear Systems, 2nd Edition. SIAM, Philadelphia, PA, 2003
27. Saad, Y., Schultz, M.H.: GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Stat. Comput. **7**, 856–869 (1986)
28. Sidi, A.: A zero-cost preconditioning for a class of indefinite linear systems. WSEAS Trans. Math. **2**, 142–150 (2003)
29. Silvester, D.J., Wathen, A.J.: Fast iterative solution of stabilised Stokes systems. II: Using general block preconditioners. SIAM J. Numer. Anal. **31**, 1352–1367 (1994)
30. Simoncini, V., Benzi, M.: Spectral properties of the Hermitian and skew-Hermitian splitting preconditioner for saddle point problems. SIAM J. Matrix Anal. Appl. **26**, 377–389 (2004)
31. Stewart, G.W.: Matrix Algorithms. Vol. II: Eigensystems, SIAM, Philadelphia, PA, 2001
32. Temam, R.: Navier–Stokes Equations, Revised Edition, Studies in Mathematics and its Applications, North-Holland, Amsterdam/New York/Oxford, 1984
33. van der Vorst, H.A.: Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of non-symmetric linear systems. SIAM J. Sci. Stat. Comput. **12**, 631–644 (1992)
34. van der Vorst, H.A.: Iterative Krylov Methods for Large Linear Systems, Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, UK, 2003
35. Wathen, A.J., Fischer, B., Silvester, D.J.: The convergence rate of the minimal residual method for the Stokes problem. Numer. Math. **71**, 121–134 (1995)