STAT 205 Probability Theory **Fall 2006**

Topic: Function Spaces

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1 Prerequisites

Random Variables, Expected Value

2 Summary

A function space is merely a set of functions, or a set of equivalence classes of functions. This document will introduce the important L^p function spaces and state their most basic properties.

3 Function Spaces

Banach spaces: Let X be a normed linear space with norm $||\cdot||$. If X is complete [1] with respect to the induced metric $d(x, y) := ||x - y||$, then the pair $(X, ||\cdot||)$ is called a *Banach space*.

Hilbert spaces: Let K be a linear space with an inner product $\langle\cdot,\cdot\rangle$. If K is complete with respect to the induced metric $d(x, y) := \sqrt{\langle x - y, x - y\rangle}$, then $(K, \langle\cdot,\cdot\rangle)$, is called a *Hilbert space*.

There are plenty of Banach spaces and Hilbert spaces that are not function spaces. For example, $\mathbb{R}^n$ equipped with the usual inner product is a Hilbert space, and hence a Banach space. Points of $\mathbb{R}^n$ (considered simply as points) are not functions, so $\mathbb{R}^n$ is not a function space. However, there is a large class of function spaces that are either Hilbert spaces or Banach spaces.

$L^p(\mu, S)$ spaces: Let $(S, \mathcal{S}, \mu)$ be a measure space. Set

$$\mathcal{L}^p(\mu, S) := \{f : S \to \mathbb{R}\colon\ f \text{ is } \mu\text{-measurable}, \int |f|^p d\mu < \infty\}.$$

[1] Recall that a metric space is complete if every Cauchy sequence converges to a limit.

When the measureable space and measure can be identified by from the context of the discussion, $\mathcal{L}^p(S, \mathcal{S}, \mu)$ is usually abreviated to $\mathcal{L}^p$. One usually places the following equivalence relation on $\mathcal{L}^p$: write $f \sim g$ iff $f, g \in \mathcal{L}^p$ and

$$\int |f - g|^p d\mu = 0.$$

This equivalence relation partitions $\mathcal{L}^p$ into equivalence classes of functions, and functions belonging to the same equivalence class are equal μ-almost everywhere. Thus we have a new space, $L^p(S, \mathcal{S}, \mu)$ consisting of equivalence classes of functions in $\mathcal{L}^p$, and we write $[f] \in L^p$ to express the assertion that an equivalence class $[f]$ is in this set. Such precision is rarely needed, however, and authors usually write f instead of $[f]$.[2]

Note that L^p is a linear space because because sums of integrable functions are integrable, and cf is integrable if f is integrable and c is a constant. Let $||[X]||_p := (\int |X|^p)^{1/p}$ be the L^p *norm* of X; this is an honest-to-goodness norm [3]. Define convergence in L^p as follows:

$$X_n \xrightarrow{L^p} X \text{ means } ||X_n - X||_p \to 0 \tag{1}$$

It can be shown that L^p is complete. Therefore L^p spaces are Banach spaces for $p \geq 1$.

L^2 is a Hilbert space if we give it the inner product $\langle f, g \rangle = \int fg \, d\mu$. You can check that the norm induced by this inner product agrees with our previous definition of the L^2 norm.

For $p = 1$ L^p corresponds with the space of integrable functions

$$L^1(\Omega, \mathcal{F}, \mathbb{P}) := \{X : X \text{ is r.v. with } \mathbb{E}(|X|) < \infty\}. \tag{2}$$

4 References

Real and Complex Analysis, Walter Rudin
Wikipedia: L^p spaces. http://en.wikipedia.org/wiki/Lp_space

[2] You do something similar every time you write $\frac{1}{2}$ instead of $[\frac{1}{2}]$, because everyone knows that a rational number $\frac{p}{q}$ is an equivalence class of ordered pairs. $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$ etc.

[3] This definition is independent of the choice of representative of the equivalence class, i.e. if $[Y] = [X]$ then we could replace X by Y in the integral above and the result would be the same.