STAT 205 Probability Theory **Fall 2006**

Topic: Change of Variables

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1 Prerequisites

Random variable, expected value, monotone convergence theorem

2 Summary

You will be presented the change of variables formula, its proof, and a simple example.

3 Change of Variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to S$ a $(\mathcal{F}\backslash\mathcal{S})$-measurable random variable. X induces a new probability measure $\mathbb{P}_X$ on $(S, \mathcal{S})$.

Definition 1 $\mathbb{P}_X(A) = \mathbb{P}(\omega : X(\omega) \in A) = \mathbb{P}(\omega : \omega = X^{-1}(A))$ *is called the* $\mathbb{P}$ *law of* X *or the* $\mathbb{P}$ *distribution of* X.

Theorem 2 (Change of variable formula) *Let* f *be measurable function from* $(S, \mathcal{S})$ *to* $(R, \mathcal{R})$. *If* $f \geq 0$ *or* $\mathbb{E}|f(X)| < \infty$ *then*

$$\mathbb{E}f(X) = \int_\Omega f(X(\omega))d\mathbb{P} = \int_S f(x)d\mathbb{P}_X$$

Proof: We will approach this proof in four different cases, each of which will build on the previous. First, for indicator functions, take $A \in \mathcal{S}$.

$$\mathbb{E}\mathbf{1}_A(X) = \mathbb{P}(X \in A) = \mathbb{P}_X(A) = \int_S \mathbf{1}_A(x)d\mathbb{P}_X$$

Second, look at simple functions. f is of the form $f(x) = \sum_{i=1}^{n} k_i \mathbf{1}_{A_i}$. Where $k_i \in \mathbb{R}$ and $A_i \in \mathcal{S}$ for $i = 1, 2, 3, ..., n$. From the linearity of integration, the case for indicator functions, and the linearity of integration, we arrive at the following equalities:

$$\mathbb{E}f(X) = \sum_{i=1}^{n} k_i \mathbb{E}\mathbf{1}_{A_i} = \sum_{i=1}^{n}[k_i \int_S \mathbf{1}_{A_i}(x) d\mathbb{P}_X] = \int_S f(x) d\mathbb{P}_X$$

Next, look at nonnegative functions f. Note: we will use $\lfloor x \rfloor$ to denote the greatest integer less than x. E.g. $\lfloor 5.3 \rfloor = 5$. Also, take $x \wedge y = \min(x, y)$. Let $f_n(x) = (\lfloor 2^n f(x) \rfloor / 2^n) \wedge n$ and notice that it is a simple function. So, $\mathbb{E}f_n(X) = \int_S f_n(x) d\mathbb{P}_X$. Additionally, $f_n \uparrow f$ and $\int_S f_n(x) d\mathbb{P}_X \uparrow \int_S f(x) d\mathbb{P}_X$ So, we can use the monotone convergence theorem.

$$\mathbb{E}f(X) = \lim_n \mathbb{E}f_n(X) = \lim_n \int_S f_n(x) d\mathbb{P}_X = \int_S f(x) d\mathbb{P}_X$$

Lastly, let's observe integrable functions. Split f into the difference between its positive and negative parts, $f(x) = f^+(x) - f^-(x)$. The condition $\mathbb{E}|f(X)| < \infty$ ensures $\mathbb{E}f^+(X) < \infty$ and $\mathbb{E}f^-(X) < \infty$. So, we can proceed by using the case of nonnegative functions and the linearity of integration.

$$\mathbb{E}f(X) = \mathbb{E}f^+(X) - \mathbb{E}f^-(X) = \int_S f^+(x) d\mathbb{P}_X - \int_S f^-(x) d\mathbb{P}_X = \int_S f(x) d\mathbb{P}_X$$

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4 References

Durrett, *Probability: Theory and Examples*, Section 1.3.