STAT 205 Probability Theory **Fall 2006**

Topic: Probability Measures

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1 Prerequisites

Basic measure theory, σ-fields, Lebesgue measure

2 Summary

This part takes concepts from measure theory to define probability measure and probability space. Then the Identitification Lemma for Probability is introduced to judge two probability measures are equal.

3 Probability Space

Definition 1 *A set function* [1] $\mathbb{P}$ *on a* σ*-field* $\mathcal{F}$ *is a* probability measure *if it satisfies the following conditions:*

1. $0 \leq \mathbb{P}(A) \leq 1$ *for* $A \in \mathcal{F}$.
2. $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$.
3. *If* $A_i \in \mathcal{F}$ *is a countable sequence of disjoint sets, then* $\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)$.

If $\mathcal{F}$ is a σ-field, then the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability measure space or simply a *probability space*. The countable additivity of the probability measure gives rise to the following properties that are stated in a theorem.

Theorem 2 *Let* $\mathbb{P}$ *be a probability measure on a field* $\mathcal{F}$.

1. *Continuity from below: If* A_n *and* A *lie in* $\mathcal{F}$ *and* $A_n \uparrow A$, *then* $\mathbb{P}(A_n) \uparrow \mathbb{P}(A)$.

[1] A set function is a real-valued function defined on some class of subsets of Ω.

2. *Continuity from above: If A_n and A lie in $\mathcal{F}$ and $A_n \downarrow A$, then $\mathbb{P}(A_n) \downarrow \mathbb{P}(A)$.*

3. *Countable subadditivity: If $A_1, A_2...$ and $\bigcup_{k=1}^{\infty} A_k$ lie in $\mathcal{F}$, then*

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mathbb{P}(A_k). \tag{1}$$

Example 3 *Let $\mathcal{R}$ = the* Borel *sets = the smallest σ-field containing the open sets. The probability space on a unit interval is then defined as $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = (0,1)$, $\mathcal{F} = \{A \cap (0,1) : A \in \mathcal{R}\}$ and $\mathbb{P}(B) = \lambda(B)$ for $B \in \mathcal{F}$. Here λ is the Lebesgue measure restricted to the Borel subsets of (0,1).*

4 Identitification Lemma for Probabilities

Now we will present a key result that would help us to extend the results we have on a field $\mathcal{A}$ to the σ-field generated by $\mathcal{A}$. This is stated as the *Identitification Lemma for Probabilities*:

Lemma 4 (Identification Lemma for Probabilities) *Let P and Q be two probability measures on $\sigma(\mathcal{A})$ where $\mathcal{A}$ is closed under intersections. If $P(A) = Q(A)$ for $A \in \mathcal{A}$, then $P(A) = Q(A)$ for all $A \in \sigma(\mathcal{A})$.*

Before we prove this lemma, let's state some basic tools from measure theory that are very useful.

Definition 5 *A collection of subsets $\mathcal{D}$ of set Ω is called a λ-*system *if*

1. *$\Omega \in \mathcal{D}$*

2. *If $A \in B$ and $B \in \mathcal{D}$, $A \subset B \Rightarrow B - A \in \mathcal{D}$*

3. *If $A_n \in \mathcal{D}$ and $A_n \uparrow A \Rightarrow A \in \mathcal{D}$*

Theorem 6 (Dynkin's π-λ Theorem) *Suppose $\mathcal{A}$ is a collection of sets closed under $\cap$ (a π-*system*). If $\mathcal{D}$ is a λ-system with $\mathcal{A} \subset \mathcal{D}$, then $\sigma(\mathcal{A}) \subset \mathcal{D}$.*

With this theorem let's try to prove the identification lemma on probabilities.

Proof: (of Lemma **??**) Consider $\mathcal{D} = \{A \in \sigma(\mathcal{A}) : P(A) = Q(A)\}$. Let's check that $\mathcal{D}$ is a λ-system.

1. $\Omega \in \mathcal{D}$ because $P(\Omega) = Q(\Omega) = 1$.
2. Let $A \in D, B \in \mathcal{D}$. Then $A \subset B$ implies $B - A \in \mathcal{D}$. This is because $P(B) = Q(B) \Rightarrow P(B-A)+P(A) = Q(B-A)+Q(A) \Rightarrow P(B-A) = Q(B-A)$.
3. $A_n \in \mathcal{D}$ and $A_n \uparrow A$ implies $A \in \mathcal{D}$. This is because $P(A_n) = Q(A_n) \Rrightarrow P(A) = Q(A)$ using theorem **??**.

Now directly applying Dynkins $\pi - \lambda$ Theorem, we get $\mathcal{A} \subset \mathcal{D}$ implies $\sigma(\mathcal{A}) \subset \mathcal{D}$. ∎

5 References

Durrett, Section 1.1