STAT 205 Probability Theory **Fall 2006**

Topic: Probability Distributions

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1 Prerequisites

One should be comfortable with basic measure theory concepts. These include the idea of a sigma field, probability measure, and probability space, $(X, \mathcal{F}, \mathbb{P})$. Additionally, knowledge of random variables is necessary. These topics are covered in *Sigma Fields*, *Probability Measures*, and *Random Variables*

2 Summary

A real-valued random variable, $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$ induces a probability measure on S. For the case of real-valued random variables this measure is called the *distribution* of X and is usually described by its *cumulative distribution function*, $F_X(x) := \mathbb{P}(X \leq x)$.

3 Probability Distributions

If $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$ is an $\mathcal{F}$-measuable r.v., then X induces a probability measure on $\mathcal{S}$. For $A \in \mathcal{S}$, set $\mu(A) := \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$. It is easy to check that μ thus defined is a probability measure. For example, observe that for countably many disjoint A_i's,

$$\mu(\cup_i A_i) = \mathbb{P}\left(X^{-1}(\cup_i A_i)\right) = \mathbb{P}\left(\cup_i X^{-1}(A_i)\right) = \sum_i \mathbb{P}\left(X^{-1}(A_i)\right) = \sum_i \mu(A_i).$$

The other properties of a probability measure can be checked in a similar manner.

In the case that the r.v. X is real-valued, we say that that the induced measure is the *distribution of X*, and describe this measure by its *cumulative distribution function* (cdf), $F_X(x) := \mathbb{P}(X \leq x)$.

Theorem 1 *A cdf F of some probability measure on $\mathbb{R}$ has the following properties:*

1. *F is a nondecreasing function of x.*
2. $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$.
3. *F is right continuous, i.e.,* $\lim_{y\downarrow x} F(y) = F(x)$.
4. *If* $F(x-) = \lim_{y\uparrow x} F(y)$ *then* $F(x-) = P(X < x)$.
5. $P(X = x) = F(x) - F(x-)$.

Proof:

1. Note that for $x \leq y$, we have $\{X \leq x\} \subset \{X \leq y\}$ so by the monotonicity of probability measures, $\mathbb{P}(X \leq x) \leq \mathbb{P}(X \leq y)$.
2. Note that as $x \uparrow \infty$, we have $\{X \leq x\} \uparrow \Omega$. Similarly, as $a \downarrow -\infty$, $\{X \leq x\} \downarrow \emptyset$. By continuity of probability measures, the desired result follows.
3. As $y \downarrow x$, we have $\{X \leq y\} \downarrow \{X \leq x\}$. By continuity of probability measures, the result follows.
4. As $y \uparrow x$, we have $\{X \leq y\} \uparrow \{X < x\}$. By continuity of probability measures, the result follows.
5. We have $\{X \leq x\} = \{X = x\} \cup \{X < x\}$. Since the events $\{X = x\}$ and $\{X < x\}$ are disjoint, this implies $\mathbb{P}(X \leq x) = \mathbb{P}(X = x) + \mathbb{P}(X < x)$. Rearranging, we have $\mathbb{P}(X = x) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x)$.

■

Theorem 2 *If F satisfies the first three properties of Theorem* **??**, *then it is the distribution function of some r.v. and there is a unique probability measure on* $(\mathbb{R}, \mathcal{R})$ *that has* $\mu((a, b]) = F(b) - F(a)$ *for all* $a, b \in \mathbb{R},\ a \leq b$.

Proof: The proof follows the proof of Durrett's Theorem 1.2. Let $F : \mathbb{R} \to (0, 1)$ have properties $1, 2, 3$ in Theorem **??**. We will construct a r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1), \mathcal{B}((0, 1)), \lambda)$, where λ denotes Lebesgue measure, and show that it has distribution function F.

Let

$$X(\omega) = \sup\{y : F(y) < \omega\}.$$

If we show that

$$\{\omega : X(\omega) \leq x\} = \{\omega : \omega \leq F(x)\}$$

then the desired result follows immediately since $\mathbb{P}(\omega : \omega \leq F(x)) = F(x)$. (Recall that λ is Lebesgue measure on (0,1) and that F is increasing, so that $\lambda\,(\omega : \omega \leq F(x)) = \sup\{\omega : \omega \leq F(x)\}$.) To check the set equality above, note that if $\omega \leq F(x)$ then $X(\omega) \leq x$, since $x \notin \{y : F(y) < \omega\}$. On the other hand, if $\omega > F(x)$, then since F is right continuous, there exists an $\epsilon > 0$ so that $F(x+\epsilon) < \omega$ and $X(\omega) \geq x+\epsilon > x$. ■

Having proved the existence of a r.v. X with distribution function F, the uniqueness can be checked by Dynkin's $\pi - \lambda$ Theorem. See Appendix A.2 in Durrett (in particular, Theorem 2.2) for further details.

4 References

Durrett, *Probability: Theory and Examples* (Third Edition), Section 1.2.