STAT 205 Probability Theory **Fall 2006**

Topic: Real Random Variables

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1 Prerequisites

Basic measure theory, random variables.

2 Summary

We give a brief introduction to real random variables, extended real random variables and simple real random variables. Extended random variables enjoy extended cumulative distribution functions, which can be used to construct compact space of distribution functions. Simple real random variable approximation, working with monotone class theorem, is a typical method in proving many equalities.

3 Real Random Variables

3.1 Checking Measurability

First, let's introduce a theorem about checking measurability, which will save our energy by reducing the verification to a smaller class of sets.

Theorem 1 *Let $(\Omega, \mathcal{F})$ be a measurable space and $X : \Omega \to S$. If S has the σ-field $\sigma(\mathcal{A})$ for an arbitrary collection of sets $\mathcal{A}$, then X is measurable iff $(X \in \mathcal{A}) \in \mathcal{F}$ for $A \in \mathcal{A}$.*

Proof: We first prove the reverse direction. Since $\{X \in A\} = \{\omega : X(\omega) \in A\} = X^{-1}(A)$, we have

$$X^{-1}(A^c) = (X^{-1}(A))^c$$
$$X^{-1}\left(\bigcup_i A_i\right) = \bigcup_i X^{-1}(A_i)$$
$$X^{-1}\left(\bigcap_i A_i\right) = \bigcap_i X^{-1}(A_i)$$

Thus, $X^{-1}(\sigma(\mathcal{A})) = \sigma(X^{-1}(\mathcal{A}))$.

To prove the forward direction, note that the collection $\mathcal{C}$ of subsets of S given by $\mathcal{C} = \{B \subset S : X^{-1}(B) \in \mathcal{F}\}$ is a σ-field which contains $\mathcal{A}$ and hence $\sigma(\mathcal{A})$ which is the σ-field generated by $\mathcal{A}$. ∎

Similarly, if S has the σ-field $\sigma(Y_i, i \in I)$, X is measurable iff each $Y_i \circ X$ is measurable.

Fact: The composition of two measurable maps is measurable.

3.2 Real Random Variables and Extended Real Random Variables

Let S be a topological space. The *Borel σ-field* on S, denoted by $\mathcal{B}(S)$, is the σ-field generated by open subsets of S. If $f : S \to T$ is a continuous function, then f is measurable from $(S, \mathcal{B}(S))$ to $(T, \mathcal{B}(T))$ by the previous theorem.

If $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{R})$, then some possible choices of $\mathcal{A}$ are $\{(-\infty, x] : x \in \mathbb{R}\}$ or $\{(-\infty, x) : x \in \mathbb{Q}\}$ where $\mathbb{Q} =$ the rationals.

For the real line $\mathbb{R} = (-\infty, \infty)$ and extended real line $\bar{\mathbb{R}} = [-\infty, \infty]$, the Borel σ-fields can be defined as follows.

$$\mathcal{B}(\mathbb{R}) = \sigma\{(-\infty, x], x \in \mathbb{R}\}$$
$$\mathcal{B}(\bar{\mathbb{R}}) = \sigma\{[-\infty, x], x \in \bar{\mathbb{R}}\}$$

Definition 2 (Real Random Variable) *Let $(\Omega, \mathcal{F})$ be a measurable space. A real random variable (r.r.v.) is a measurable map from Ω to $\mathbb{R}$.*

Thus a function X with range $\mathbb{R}$ is a r.v. iff $(X \leq x) \in \mathcal{F}$ for all $x \in \mathbb{R}$ (by theorem 2.1). Similarly, extended real random variables (e.r.r.v.) can be defined on range $\bar{\mathbb{R}}$.

Operations on real numbers are performed pointwise on real-valued functions, e.g.,

$$Z = X + Y \text{ means } Z(\omega) = X(\omega) + Y(\omega) \text{ for all } \omega \in \Omega$$
$$\text{and } Z = \lim_n Z_n \text{ means } Z(\omega) = \lim_n Z_n(\omega) \text{ for all } \omega \in \Omega$$

Notation for real numbers: $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$, $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$. Note that $|x| = x^+ + x^-$ and $x = x^+ - x^-$.

Theorem 3 *If $X_1, X_2, \ldots$ are e.r.r.v.'s on $(\Omega, \mathcal{F})$, then they are closed under all limiting operations, i.e.,*

$$\inf_n X_n, \ \sup_n X_n, \ \liminf_n X_n, \ \limsup_n X_n$$

are also e.r.r.v.

Proof: Since the infimum of a sequence is $< a$ iff some term is $< a$, we have

$$\left\{\inf_n X_n < a\right\} = \bigcup_n \{X_n < a\} \in \mathcal{F}$$

The proof for supremum follows similarly.

For limit inferior of X_n, we have

$$\liminf_{n\to\infty} X_n := \sup_n \{\inf_{m\geq n} X_m\}$$

Now note that $Y_n = \inf_{m\geq n} X_m$ is an e.r.r.v. for each n and so $\sup_n Y_n$ is also an e.r.r.v. The proof for limit superior follows similarly. ■

From the above proof we see that

$$\Omega_0 \equiv \left\{\omega : \lim_{n\to\infty} X_n \text{ exists } \right\} = \left\{\omega : \limsup_{n\to\infty} X_n - \liminf_{n\to\infty} X_n = 0\right\}$$

is a measurable set. If $X_n(\omega)$ converges for almost all ω, i.e., $\mathbb{P}(\Omega_0) = 1$, we say that X_n *converges almost surely* to a limit X which is defined on Ω_0. X can be defined arbitrarily on $\Omega \setminus \Omega_0$, with different authors preferring different conventions.

3.3 Simple Real Random Variables

Definition 4 (Simple Random Variable) *X is a simple random variable iff X is a finite linear combination of indicators, i.e., X can be expressed as $X(\omega) = \sum_{i=1}^n c_i 1_{A_i}(\omega)$ where $c_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$. A simple r.v. can only take finitely many values.*

Theorem 5 *Every real r.v. X is a pointwise limit of a sequence of simple r.v.'s, which can be taken to be increasing if $X \geq 0$.*

Proof: For $X \geq 0$ let,

$$X_n = \begin{cases} \frac{k-1}{2^n} & \text{on } \{\frac{k-1}{2^n} \leq X < \frac{k}{2^n}\}, 0 \leq k \leq n2^n \\ n & \text{on } \{X \geq n\} \end{cases}$$

Then $X_n \uparrow X$. For general X use the decomposition $X = X^+ - X^-$. ■

Corollary 6 *Let X and Y be real valued r.v.'s. Then so are XY, $X+Y$, $X-Y$, $\min(X,Y)$, $\max(X,Y)$.*

Proof: Consider $X_n \uparrow X$ and $Y_n \uparrow Y$. This implies $X_nY_n \uparrow XY$. Similarly, use the previous theorem to pass from simple case to the more general cases. ■

4 References

Durrett, *Probability: Theory and Examples* (Third Edition), Section 1.2.